19. XI 2018.

WICKOVA TEOREMA NA KONAČNOJ TEMPERATURI

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* izračunavanje srednje vrednosti proizvoda operatora kreacije i anihilacije u ravnotežnom stanju idealnog gasa (veliki kanonski ansambl)

$$\hat{\mathcal{H}}_0 = \sum_f \varepsilon_f \hat{a}_f^\dagger \hat{a}_f \ , \quad \hat{\varrho}_0 = \frac{1}{\Xi} e^{-\beta(\hat{\mathcal{H}}_0 - \mu\hat{\mathcal{N}})} = e^{\beta(\Omega - (\hat{\mathcal{H}}_0 - \mu\hat{\mathcal{N}}))} \ , \quad \hat{\mathcal{N}} = \sum_f \hat{a}_f^\dagger \hat{a}_f$$

$$\Xi = \mathrm{Tr}\left(e^{-\beta(\hat{\mathcal{H}}_0 - \mu\hat{\mathcal{N}})}\right) \ , \quad \Omega = -k_B T \ln \Xi$$

* termodinamička srednja vrednost operatora $\hat{A}$ (veliki kanonski ansambl, idealni gas) $\langle \hat{A} \rangle_0 = \mathrm{Tr}(\hat{\varrho}_0 \hat{A})$
* $[\hat{\mathcal{H}}_0, \hat{N}] = 0$, $\quad [\hat{\varrho}_0, \hat{N}] = 0$
* neka je $\hat{A} = \hat{a}^\dagger_{f_1} \dots \hat{a}^\dagger_{f_n} \hat{a}_{f_1'} \dots \hat{a}_{f_m'}$ i neka je unitarni operator $\hat{U}$ definisan kao $\hat{U} = e^{i\alpha\hat{\mathcal{N}}}$, $\alpha \in \mathbb{R}$

$$\langle \hat{A} \rangle_0 = \mathrm{Tr}\Big(\underset{\hat{U}^\dagger\hat{U}}{\wedge}\hat{a}^\dagger_{f_1} \underset{\hat{U}^\dagger\hat{U}}{\wedge} \dots \hat{a}^\dagger_{f_n} \underset{\hat{U}^\dagger\hat{U}}{\wedge} \hat{a}_{f_1'} \underset{\hat{U}^\dagger\hat{U}}{\wedge} \dots \hat{a}_{f_m'} \underset{\hat{U}^\dagger\hat{U}}{\wedge} \hat{\varrho}_0\Big) = \mathrm{Tr}\Big(\hat{U}^\dagger \boxed{\hat{U}\hat{a}^\dagger_{f_1}\hat{U}^\dagger}\boxed{\hat{U}\hat{a}^\dagger_{f_2}\hat{U}^\dagger}\hat{U} \dots \hat{U}^\dagger \boxed{\hat{U}\hat{a}^\dagger_{f_n}\hat{U}^\dagger}\boxed{\hat{U}\hat{a}_{f_1'}\hat{U}^\dagger}\hat{U} \dots \hat{U}^\dagger \boxed{\hat{U}\hat{a}_{f_m'}\hat{U}^\dagger}\hat{U}\hat{\varrho}_0\Big)$$

$$\hat{U}\hat{a}^\dagger_{f_1}\hat{U}^\dagger = e^{i\alpha\hat{\mathcal{N}}}\hat{a}^\dagger_{f_1} e^{-i\alpha\hat{\mathcal{N}}} = \hat{a}^\dagger_{f_1} + [i\alpha\hat{\mathcal{N}}, \hat{a}^\dagger_{f_1}] + \frac{1}{2!}[i\alpha\hat{\mathcal{N}}, [i\alpha\hat{\mathcal{N}}, \hat{a}^\dagger_{f_1}]] + \dots$$

$$[\hat{\mathcal{N}}, \hat{a}^\dagger_{f_1}] = \sum_f [\hat{a}^\dagger_f \hat{a}_f, \hat{a}^\dagger_{f_1}] = \begin{cases} \text{bozoni} & \sum_f \hat{a}^\dagger_f [\hat{a}_f, \hat{a}^\dagger_{f_1}] \\ \text{fermioni} & \sum_f \hat{a}^\dagger_f \{\hat{a}_f, \hat{a}^\dagger_{f_1}\} \end{cases} = \sum_f \hat{a}^\dagger_f \delta_{ff_1} = \hat{a}^\dagger_{f_1}$$

$$[\hat{\mathcal{N}}, [\hat{\mathcal{N}}, \hat{a}^\dagger_{f_1}]] = [\hat{\mathcal{N}}, \hat{a}^\dagger_{f_1}] = \hat{a}^\dagger_{f_1} \quad \text{itd.}$$

$$\Rightarrow \hat{U}\hat{a}^\dagger_{f_1}\hat{U}^\dagger = \hat{a}^\dagger_{f_1} + i\alpha\hat{a}^\dagger_{f_1} + \frac{(i\alpha)^2}{2!}\hat{a}^\dagger_{f_1} + \dots \Rightarrow \boxed{\hat{U}\hat{a}^\dagger_{f_1}\hat{U}^\dagger = e^{i\alpha}\hat{a}^\dagger_{f_1}} \quad \boxed{\hat{U}\hat{a}_{f_1'}\hat{U}^\dagger = e^{-i\alpha}\hat{a}_{f_1'}}$$

$$\begin{aligned}
\langle \hat{A} \rangle_0 &= \mathrm{Tr}\left(\hat{U}^\dagger e^{i\alpha}\hat{a}^\dagger_{f_1} \dots e^{i\alpha}\hat{a}^\dagger_{f_n} e^{-i\alpha}\hat{a}_{f_1'} \dots e^{-i\alpha}\hat{a}_{f_m'}\hat{U}\hat{\varrho}_0\right) \\
&= e^{i\alpha n} e^{-i\alpha m}\, \mathrm{Tr}\left(\hat{U}^\dagger \hat{a}^\dagger_{f_1} \dots \hat{a}^\dagger_{f_n} \hat{a}_{f_1'} \dots \hat{a}_{f_m'}\hat{U}\hat{\varrho}_0\right) \\
&= e^{i\alpha(n-m)}\, \mathrm{Tr}\left(\hat{a}^\dagger_{f_1} \dots \hat{a}^\dagger_{f_n} \hat{a}_{f_1'} \dots \hat{a}_{f_m'}\hat{U}\hat{\varrho}_0\hat{U}^\dagger\right) \qquad \text{cikličnost pod tragom} \\
&= e^{i\alpha(n-m)}\, \mathrm{Tr}\left(\hat{a}^\dagger_{f_1} \dots \hat{a}^\dagger_{f_n} \hat{a}_{f_1'} \dots \hat{a}_{f_m'}\hat{\varrho}_0\right) \qquad [\hat{\varrho}_0, \hat{\mathcal{N}}] = 0 \\
&= e^{i\alpha(n-m)} \langle \hat{A} \rangle_0
\end{aligned}$$

$$\boxed{\langle \hat{A} \rangle_0 \left(1 - e^{i\alpha(n-m)}\right) = 0}$$ ova jednakost je tačna za proizvoljno $\alpha \in \mathbb{R}$

* ako je $\langle \hat{A} \rangle_0 \neq 0$, onda jednakost $1 - e^{i\alpha(n-m)} = 0$ mora važiti za proizvoljno $\alpha \in \mathbb{R}$, što je moguće jedino za $n = m$
* ako je $n \neq m$, onda je, u opštem slučaju, $1 - e^{i\alpha(n-m)} \neq 0$, pa da bi jednakost $\langle \hat{A} \rangle_0 (1 - e^{i\alpha(n-m)}) = 0$ važila za proizvoljno $\alpha \in \mathbb{R}$, mora biti $\langle \hat{A} \rangle_0 = 0$
* primetite da poredak operatora kreacije i anihilacije u operatoru $\hat{A}$ ne utiče na izvedena tvrđenja; u gornjem izvođenju smo, određenosti radi, fiksirali normalni poredak (svi kreacioni operatori stoje levo od anihilacionih operatora)

$\Rightarrow$ (1) srednja vrednost proizvoda nejednakog broja kreacionih i anihilacionih operatora po velikom kanonskom ansamblu za idealni gas je jednaka nuli

(2) srednja vrednost proizvoda neparnog broja kreacionih i anihilacionih operatora po velikom kanonskom ansamblu za idealni gas je jednaka nuli

(3) srednja vrednost proizvoda kreacionih i anihilacionih operatora po velikom kanonskom ansamblu za idealni gas može biti nenulta jedino ako je broj kreacionih operatora jednak broju anihilacionih operatora

* primeri $\langle \hat a^\dagger_{f_1} \hat a^\dagger_{f_2}\rangle_0 = 0$, $\langle \hat a_{f_1} \hat a_{f_2}\rangle_0 = 0$

$$\langle \hat a^\dagger_{f_1} \hat a_{f_2}\rangle_0 = \frac{1}{\Xi}\operatorname{Tr}\left(\hat a^\dagger_{f_1}\hat a_{f_2}\, e^{-\beta\sum_f(\varepsilon_f-\mu)\hat a^\dagger_f \hat a_f}\right)$$
$$= \frac{1}{\Xi}\operatorname{Tr}\left(\hat a^\dagger_{f_1}\hat a_{f_2}\prod_f e^{-\beta(\varepsilon_f-\mu)\hat a^\dagger_f\hat a_f}\right)$$ pošto svaka dva sabirka u eksponentu komutiraju

pomoćno tvrđenje $e^{\lambda \hat a^\dagger_f \hat a_f}\,\hat a_{f_2}\, e^{-\lambda \hat a^\dagger_f\hat a_f} = \hat a_{f_2} + \lambda[\hat a^\dagger_f\hat a_f, \hat a_{f_2}] + \frac{\lambda^2}{2!}[\hat a^\dagger_f \hat a_f, [\hat a^\dagger_f\hat a_f, \hat a_{f_2}]] + \dots$, $\lambda\in\mathbb{R}$

$$[\hat a^\dagger_f\hat a_f, \hat a_{f_2}] = \begin{cases}\text{bozoni} & [\hat a^\dagger_f, \hat a_{f_2}]\hat a_f \\ \text{fermioni} & -\{\hat a^\dagger_f, \hat a_{f_2}\}\hat a_f\end{cases} = -\delta_{ff_2}\hat a_{f_2}$$

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$$e^{\lambda \hat a^\dagger_f \hat a_f}\,\hat a_{f_2}\, e^{-\lambda \hat a^\dagger_f\hat a_f} = \hat a_{f_2} - \delta_{ff_2}\lambda\hat a_{f_2} + \delta_{ff_2}\frac{\lambda^2}{2!}\hat a_{f_2} + \dots$$
$$= (1-\delta_{ff_2})\hat a_{f_2} + \delta_{ff_2}\left(1-\lambda+\frac{\lambda^2}{2!}-\dots\right)\hat a_{f_2} = (1-\delta_{ff_2})\hat a_{f_2} + \delta_{ff_2}e^{-\lambda}\hat a_{f_2}$$

$$\boxed{e^{\lambda \hat a^\dagger_f \hat a_f}\,\hat a_{f_2}\, e^{-\lambda \hat a^\dagger_f\hat a_f} = \begin{cases}\hat a_{f_2}, & f\neq f_2\\ e^{-\lambda}\hat a_{f_2}, & f=f_2\end{cases}}$$

$$\boxed{\hat a_{f_2}\, e^{-\lambda \hat a^\dagger_f\hat a_f} = \begin{cases} e^{-\lambda \hat a^\dagger_f\hat a_f}\hat a_{f_2}, & f\neq f_2\\ e^{-\lambda}\, e^{-\lambda \hat a^\dagger_f\hat a_f}\hat a_{f_2}, & f=f_2\end{cases}}$$

$$\boxed{\hat a_{f_2}\prod_f e^{-\beta(\varepsilon_f-\mu)\hat a^\dagger_f\hat a_f} = e^{-\beta(\varepsilon_{f_2}-\mu)}\prod_f e^{-\beta(\varepsilon_f-\mu)\hat a^\dagger_f\hat a_f}\,\hat a_{f_2}}$$

$$\langle \hat a^\dagger_{f_1}\hat a_{f_2}\rangle_0 = e^{-\beta(\varepsilon_{f_2}-\mu)}\frac{1}{\Xi}\operatorname{Tr}\left(\hat a^\dagger_{f_1}\prod_f e^{-\beta(\varepsilon_f-\mu)\hat a^\dagger_f\hat a_f}\,\hat a_{f_2}\right)$$
$$= e^{-\beta(\varepsilon_{f_2}-\mu)}\operatorname{Tr}\left(\hat a_{f_2}\hat a^\dagger_{f_1}\hat\varrho_0\right)$$ ciklićnost pod tragom
$$= e^{-\beta(\varepsilon_{f_2}-\mu)}\operatorname{Tr}\left((\delta_{f_1f_2}\pm \hat a^\dagger_{f_1}\hat a_{f_2})\hat\varrho_0\right)$$ gornji znak: bozoni, donji znak: fermioni
$$= e^{-\beta(\varepsilon_{f_2}-\mu)}\left(\delta_{f_1f_2}\pm\langle \hat a^\dagger_{f_1}\hat a_{f_2}\rangle_0\right)$$

$$\Rightarrow \langle \hat a^\dagger_{f_1}\hat a_{f_2}\rangle_0 = \frac{e^{-\beta(\varepsilon_{f_2}-\mu)}}{1\mp e^{-\beta(\varepsilon_{f_2}-\mu)}}\delta_{f_1f_2}\,, \quad \boxed{\langle \hat a^\dagger_{f_1}\hat a_{f_2}\rangle_0 = \delta_{f_1f_2}\frac{1}{e^{\beta(\varepsilon_{f_2}-\mu)}\mp 1}} = \delta_{f_1f_2}\bar n_{f_2}$$

$$\langle \hat a_{f_1}\hat a^\dagger_{f_2}\rangle_0 = \langle \delta_{f_1f_2}\pm \hat a^\dagger_{f_2}\hat a_{f_1}\rangle_0 = \delta_{f_1f_2}\pm\delta_{f_1f_2}\frac{1}{e^{\beta(\varepsilon_{f_1}-\mu)}\mp 1}$$

$$\boxed{\langle \hat a_{f_1}\hat a^\dagger_{f_2}\rangle_0 = \delta_{f_1f_2}(1\pm\bar n_{f_1})}$$

* razmotrimo sada opšti slučaj, $\langle \hat\gamma_{f_1}\hat\gamma_{f_2}\cdots\hat\gamma_{f_{2n}}\rangle_0 = ?$
srednje vrednosti proizvoda parnog broja ($2n$) operatora kreacije i anihilacije
* oznaka $\hat\gamma_f$ označava ili kreacioni ($\hat a^\dagger_f$) ili anihilacioni ($\hat a_f$) operator
* određenosti radi, u nastavku ćemo smatrati da operatori $\hat a_f, \hat a^\dagger_f$ zadovoljavaju fermionsku antikomutacionu algebru; rezonovanje u slučaju bozonske komutacione algebre je analogno

$$\hat\gamma_{f_1}\hat\gamma_{f_2}\hat\gamma_{f_3}\cdots\hat\gamma_{f_{2n}} = \left(\{\hat\gamma_{f_1},\hat\gamma_{f_2}\} - \hat\gamma_{f_2}\hat\gamma_{f_1}\right)\hat\gamma_{f_3}\cdots\hat\gamma_{f_{2n}}$$
$$= \{\hat\gamma_{f_1},\hat\gamma_{f_2}\}\hat\gamma_{f_3}\cdots\hat\gamma_{f_{2n}} - \hat\gamma_{f_2}\left(\{\hat\gamma_{f_1},\hat\gamma_{f_3}\} - \hat\gamma_{f_3}\hat\gamma_{f_1}\right)\hat\gamma_{f_4}\cdots\hat\gamma_{f_{2n}}$$
$$= \{\hat\gamma_{f_1},\hat\gamma_{f_2}\}\hat\gamma_{f_3}\cdots\hat\gamma_{f_{2n}} - \{\hat\gamma_{f_1},\hat\gamma_{f_3}\}\hat\gamma_{f_2}\hat\gamma_{f_4}\cdots\hat\gamma_{f_{2n}} + \hat\gamma_{f_2}\hat\gamma_{f_3}\hat\gamma_{f_1}\hat\gamma_{f_4}\cdots\hat\gamma_{f_{2n}}$$

ovde smo iskoristili činjenicu da su antikomutatori $\{\hat\gamma_{f_1},\hat\gamma_{f_3}\}$ zapravo c-brojevi

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pomerajući tako operator $\hat{\gamma}_{f_1}$ sve do kraja niza, dobijamo

$$\boxed{\begin{aligned}\hat{\gamma}_{f_1}\hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n}} &= \{\hat{\gamma}_{f_1},\hat{\gamma}_{f_2}\}\,\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n}}\\ &- \{\hat{\gamma}_{f_1},\hat{\gamma}_{f_3}\}\,\hat{\gamma}_{f_2}\hat{\gamma}_{f_4}\cdots\hat{\gamma}_{f_{2n}}\\ &+ \{\hat{\gamma}_{f_1},\hat{\gamma}_{f_4}\}\,\hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\hat{\gamma}_{f_5}\cdots\hat{\gamma}_{f_{2n}}\\ &- \cdots + \{\hat{\gamma}_{f_1},\hat{\gamma}_{f_{2n}}\}\,\hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n-1}} - \hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n}}\hat{\gamma}_{f_1}\end{aligned}}$$

* pitanje: oznaka $\hat{\gamma}_f$, kao i pretpostavka o fermionskoj antikomutacionoj algebri, bi trebalo da vas podsete na slično izvođenje koje ste videli u drugom kontekstu – na šta vas podseća?

$$\langle\hat{\gamma}_{f_1}\hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n}}\rangle_0 = \{\hat{\gamma}_{f_1},\hat{\gamma}_{f_2}\}\langle\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n}}\rangle_0 - \{\hat{\gamma}_{f_1},\hat{\gamma}_{f_3}\}_0\langle\hat{\gamma}_{f_2}\hat{\gamma}_{f_4}\cdots\hat{\gamma}_{f_{2n}}\rangle_0 + \cdots$$
$$+\{\hat{\gamma}_{f_1},\hat{\gamma}_{f_{2n}}\}\langle\hat{\gamma}_{f_2}\cdots\hat{\gamma}_{f_{2n-1}}\rangle_0 - \langle\hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n}}\hat{\gamma}_{f_1}\rangle_0$$

$$\langle\hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n}}\hat{\gamma}_{f_1}\rangle_0 = \frac{1}{\Xi}\,\mathrm{Tr}\Big(\hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n}}\hat{\gamma}_{f_1}\prod_f e^{-\beta(\varepsilon_f-\mu)\hat{a}_f^\dagger\hat{a}_f}\Big)$$

ako je $\hat{\gamma}_{f_1} = \hat{a}_{f_1}$, već smo izveli da je $\quad \hat{a}_{f_1}\prod_f e^{-\beta(\varepsilon_f-\mu)\hat{a}_f^\dagger\hat{a}_f} = e^{-\beta(\varepsilon_{f_1}-\mu)}\prod_f e^{-\beta(\varepsilon_f-\mu)\hat{a}_f^\dagger\hat{a}_f}\,\hat{a}_{f_1}$

ako je $\hat{\gamma}_{f_1} = \hat{a}_{f_1}^\dagger$, možete pokazati da je $\quad \hat{a}_{f_1}^\dagger\prod_f e^{-\beta(\varepsilon_f-\mu)\hat{a}_f^\dagger\hat{a}_f} = e^{\beta(\varepsilon_{f_1}-\mu)}\prod_f e^{-\beta(\varepsilon_f-\mu)\hat{a}_f^\dagger\hat{a}_f}\,\hat{a}_{f_1}^\dagger$

$$\Rightarrow\quad \langle\hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n}}\hat{\gamma}_{f_1}\rangle_0 = e^{\mp\beta(\varepsilon_{f_1}-\mu)}\,\mathrm{Tr}\big(\hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n}}\hat{\varsigma}_0\hat{\gamma}_{f_1}\big)$$

gornji znak $\hat{\gamma}_{f_1} = \hat{a}_{f_1}$
donji znak $\hat{\gamma}_{f_1} = \hat{a}_{f_1}^\dagger$

$$= e^{\mp\beta(\varepsilon_{f_1}-\mu)}\,\mathrm{Tr}\big(\hat{\gamma}_{f_1}\hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n}}\hat{\varsigma}_0\big)$$

cikličnost pod tragom

$$= e^{\mp\beta(\varepsilon_{f_1}-\mu)}\,\langle\hat{\gamma}_{f_1}\hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n}}\rangle_0$$

$$\Rightarrow\quad \boxed{\begin{aligned}\langle\hat{\gamma}_{f_1}\hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n}}\rangle_0 &= \frac{\{\hat{\gamma}_{f_1},\hat{\gamma}_{f_2}\}}{1+e^{\mp\beta(\varepsilon_{f_1}-\mu)}}\langle\hat{\gamma}_{f_3}\hat{\gamma}_{f_4}\cdots\hat{\gamma}_{f_{2n}}\rangle_0\\ &- \frac{\{\hat{\gamma}_{f_1},\hat{\gamma}_{f_3}\}}{1+e^{\mp\beta(\varepsilon_{f_1}-\mu)}}\langle\hat{\gamma}_{f_2}\hat{\gamma}_{f_4}\cdots\hat{\gamma}_{f_{2n}}\rangle_0 + \cdots + \frac{\{\hat{\gamma}_{f_1},\hat{\gamma}_{f_{2n}}\}}{1+e^{\mp\beta(\varepsilon_{f_1}-\mu)}}\langle\hat{\gamma}_{f_2}\hat{\gamma}_{f_3}\cdots\hat{\gamma}_{f_{2n-1}}\rangle_0\end{aligned}}$$

$\dfrac{\{\hat{\gamma}_{f_1},\hat{\gamma}_{f_2}\}}{1+e^{\mp\beta(\varepsilon_{f_1}-\mu)}}$	$\hat{\gamma}_{f_2} = \hat{a}_{f_2}$	$\hat{\gamma}_{f_2} = \hat{a}_{f_2}^\dagger$
$\hat{\gamma}_{f_1} = \hat{a}_{f_1}$	$\emptyset$	$\dfrac{\delta_{f_1f_2}}{1+e^{-\beta(\varepsilon_{f_1}-\mu)}} = \delta_{f_1f_2}\dfrac{e^{\beta(\varepsilon_{f_1}-\mu)}}{e^{\beta(\varepsilon_{f_1}-\mu)}+1} = \delta_{f_1f_2}(1-\bar{n}_{f_1})$
$\hat{\gamma}_{f_1} = \hat{a}_{f_1}^\dagger$	$\dfrac{\delta_{f_1f_2}}{1+e^{\beta(\varepsilon_{f_1}-\mu)}} = \delta_{f_1f_2}\bar{n}_{f_1}$	$\emptyset$

$$\Rightarrow\quad \boxed{\frac{\{\hat{\gamma}_{f_1},\hat{\gamma}_{f_2}\}}{1+e^{\mp\beta(\varepsilon_{f_1}-\mu)}} = \langle\hat{\gamma}_{f_1}\hat{\gamma}_{f_2}\rangle_0}$$

tako da smo srednju vrednost proizvoda $2n$ operatora kreacije i anihilacije uspeli da prikažemo kao sumu proizvoda srednje vrednosti 2 operatora i srednje vrednosti preostalih $(2n-2)$ operatora; svaki član u sumi ulazi sa predznakom + ili – u zavisnosti od toga da li je broj razmena fermionskih operatora neophodnih da se formira par (2 operatora jedan do drugog) paran ili neparan

$$\boxed{\langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_2} \hat{\gamma}_{f_3} \cdots \hat{\gamma}_{f_{2n}} \rangle_0 = \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_2} \rangle_0 \langle \hat{\gamma}_{f_3} \cdots \hat{\gamma}_{f_{2n}} \rangle_0 - \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_3} \rangle_0 \langle \hat{\gamma}_{f_2} \hat{\gamma}_{f_4} \cdots \hat{\gamma}_{f_{2n}} \rangle_0 + \ldots + \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_{2n}} \rangle_0 \langle \hat{\gamma}_{f_2} \cdots \hat{\gamma}_{f_{2n-1}} \rangle_0}$$

fermioni

za bozone, analogno tvrđenje bi glasilo

$$\boxed{\langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_2} \hat{\gamma}_{f_3} \cdots \hat{\gamma}_{f_{2n}} \rangle_0 = \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_2} \rangle_0 \langle \hat{\gamma}_{f_3} \cdots \hat{\gamma}_{f_{2n}} \rangle_0 + \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_3} \rangle_0 \langle \hat{\gamma}_{f_2} \hat{\gamma}_{f_4} \cdots \hat{\gamma}_{f_{2n}} \rangle_0 + \ldots + \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_{2n}} \rangle_0 \langle \hat{\gamma}_{f_2} \cdots \hat{\gamma}_{f_{2n-1}} \rangle_0}$$

bozoni

postupak bismo dalje mogli da nastavimo tako što bismo srednje vrednosti proizvoda $(2n-2)$ operatora kreacije i anihilacije predstavili kao sumu proizvoda srednje vrednosti 2 operatora i srednje vrednosti preostalih $(2n-4)$ operatora, itd., sve dok ne bismo sve izrazili kao sumu proizvoda srednjih vrednosti parova

* primer

$$\langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_2} \hat{\gamma}_{f_3} \hat{\gamma}_{f_4} \rangle_0 = \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_2} \rangle_0 \langle \hat{\gamma}_{f_3} \hat{\gamma}_{f_4} \rangle_0 \pm \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_3} \rangle_0 \langle \hat{\gamma}_{f_2} \hat{\gamma}_{f_4} \rangle + \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_4} \rangle_0 \langle \hat{\gamma}_{f_2} \hat{\gamma}_{f_3} \rangle_0$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ | & | & | & | \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad \begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{matrix}$$

neparna permutacija
$\Rightarrow$ znak $-$ za fermione

parna permutacija

$$\begin{aligned} \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_2} \hat{\gamma}_{f_3} \hat{\gamma}_{f_4} \hat{\gamma}_{f_5} \hat{\gamma}_{f_6} \rangle_0 = & \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_2} \rangle_0 \langle \hat{\gamma}_{f_3} \hat{\gamma}_{f_4} \hat{\gamma}_{f_5} \hat{\gamma}_{f_6} \rangle_0 \\ & \pm \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_3} \rangle_0 \langle \hat{\gamma}_{f_2} \hat{\gamma}_{f_4} \hat{\gamma}_{f_5} \hat{\gamma}_{f_6} \rangle_0 \\ & + \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_4} \rangle_0 \langle \hat{\gamma}_{f_2} \hat{\gamma}_{f_3} \hat{\gamma}_{f_5} \hat{\gamma}_{f_6} \rangle_0 \\ & \pm \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_5} \rangle_0 \langle \hat{\gamma}_{f_2} \hat{\gamma}_{f_3} \hat{\gamma}_{f_4} \hat{\gamma}_{f_6} \rangle_0 \\ & + \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_6} \rangle_0 \langle \hat{\gamma}_{f_2} \hat{\gamma}_{f_3} \hat{\gamma}_{f_4} \hat{\gamma}_{f_5} \rangle_0 \end{aligned}$$

$$\begin{aligned} = & \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_2} \rangle_0 \langle \hat{\gamma}_{f_3} \hat{\gamma}_{f_4} \rangle_0 \langle \hat{\gamma}_{f_5} \hat{\gamma}_{f_6} \rangle_0 \pm \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_2} \rangle_0 \langle \hat{\gamma}_{f_3} \hat{\gamma}_{f_5} \rangle_0 \langle \hat{\gamma}_{f_4} \hat{\gamma}_{f_6} \rangle_0 + \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_2} \rangle_0 \langle \hat{\gamma}_{f_3} \hat{\gamma}_{f_6} \rangle_0 \langle \hat{\gamma}_{f_4} \hat{\gamma}_{f_5} \rangle_0 \\ & \pm \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_3} \rangle_0 \langle \hat{\gamma}_{f_2} \hat{\gamma}_{f_4} \rangle_0 \langle \hat{\gamma}_{f_5} \hat{\gamma}_{f_6} \rangle_0 \mp \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_3} \rangle_0 \langle \hat{\gamma}_{f_2} \hat{\gamma}_{f_5} \rangle_0 \langle \hat{\gamma}_{f_4} \hat{\gamma}_{f_6} \rangle_0 \pm \langle \hat{\gamma}_{f_1} \hat{\gamma}_{f_3} \rangle_0 \langle \hat{\gamma}_{f_2} \hat{\gamma}_{f_6} \rangle \langle \hat{\gamma}_{f_4} \hat{\gamma}_{f_5} \rangle_0 \\ & + \text{još } 9 \text{ sabiraka} \end{aligned}$$

svaki sabirak ulazi sa predznakom (za fermione!) + ili − u zavisnosti od toga da li je broj razmena fermionskih operatora neophodnih da se operatori u parovima dovedu jedan do drugog paran ili neparan

* Koliko ima sabiraka?

$\binom{2n}{2}$ = broj načina da odaberem 1. par

$\binom{2n-2}{2}$ = broj načina da odaberem 2. par

$\vdots$

$\binom{2}{2} = 1$ = broj načina da odaberem n-ti par

poredak parova nije važan!
($n!$ permutacija od n parova daje isti sabirak)

$$\Rightarrow \boxed{\text{broj sabiraka}} = \frac{1}{n!} \binom{2n}{2} \binom{2n-2}{2} \cdots \binom{2}{2} = \frac{1}{n!} \frac{(2n)!}{2!\,\cancel{(2n-2)!}} \frac{\cancel{(2n-2)!}}{2!\,\cancel{(2n-4)!}} \cdots \frac{\cancel{2!}}{2!\,0!}$$

$$= \frac{1}{n!} \frac{(2n)!}{2^n} = \frac{1}{n!} \frac{2n(2n-1)(2n-2)\ldots 2\cdot 1}{2^n} = \frac{2\cdot n \cdot 2 \cdot (n-1) \ldots 2}{2 \cdot n \cdot 2 (n-1) \ldots 2} (2n-1)(2n-3) \cdots 1$$

$$= \boxed{(2n-1)!!}$$

$n = 2 \Rightarrow (2\cdot 2-1)!! = 3$ sabirka

$n = 3 \Rightarrow (2\cdot 3-1)!! = 5\cdot 3 = 15$ sabiraka