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Lattice-like structures in Lowest Landau Level

Master's thesis

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Motivation

The main idea of this thesis is to construct an effective Hamiltonian for a bosonic system on a square lattice, that will support FQHE at $\nu = \frac{1}{2}$. The goal is to realize the bosonic Laughlin state without any complex gauge fields or an external magnetic field. This realization should be accomplished through inner site interactions. We are working in an anisotropic gauge to identify the key elements responsible for the effect. The final model Hamiltonian would be an anisotropic one resembling a wire construction scheme.[1] We start with the LLL wave functions on a torus in the Landau (anisotropic) gauge and use the basis of Wu et al.[2] to construct anisotropic magnetic Wannier functions and express the density operator. Following this, the essential elements of the interaction operator are extracted in a long wavelength approximation. A numerical study of magnetic Wannier wave functions is presented as well as an analysis of interaction operator key elements responsible for the effect.

Chapter 1

Introduction

1.1 Particle in a magnetic field

From quantum mechanics we know that the Hamiltonian for a free particle is:

$$\hat{H} = \frac{\mathbf{p}^2}{2m}$$

where $\mathbf{p}$ is the momentum operator, and m is the mass of a particle. When a particle is subjected to a magnetic field in order to write the Hamiltonian for that system we need to replace the free-particle momentum with gauge invariant one, the procedure called *minimal substitution*.

$$\mathbf{p} \to \mathbf{\Pi} = \mathbf{p} + e\mathbf{A}(\mathbf{r}) \tag{1.1.1}$$

where $\mathbf{A}(\mathbf{r})$ is the vector potential of the given magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r})$. This new gauge invariant momentum is proportional to the particle velocity which is a gauge invariant as a physical quantity. We see that vector potential is not gauge invariant so neither is the momentum $\mathbf{p}$. Using a property of a curl we see that adding a gradient of an arbitrary function to the vector potential will leave a magnetic field unchanged $\mathbf{A}(\mathbf{r}) \to \mathbf{A}(\mathbf{r}) + \boldsymbol{\nabla} f(\mathbf{r})$. From this we can see, in order for $\mathbf{\Pi}$ to stay gauge invariant the momentum has to transform like $\mathbf{p} \to \mathbf{p} - e\boldsymbol{\nabla} f(\mathbf{r})$. The new Hamiltonian for a particle in a magnetic filed will be:

$$\hat{H}_B = \frac{(\mathbf{p} + e\mathbf{A}(\mathbf{r}))^2}{2m} \tag{1.1.2}$$

Further, we will only assume that the particle is bounded to a plane and can only move in 2D.

1.1.1 Gauge invariant momentum

In the last section we have introduced the gauge invariant momentum $\mathbf{\Pi}$, let us now find the commutation relations between the different components of the momentum:

$$[\Pi_x, \Pi_y] = [p_x + eA_x(\mathbf{r}), p_y + eA_y(\mathbf{r})] = e([p_x, A_y] - [p_y, A_x]) \tag{1.1.3}$$

we can now use the relation $[A, f(B)] = \frac{\mathrm{d}f}{\mathrm{d}B}[A, B]$ to get

$$[\Pi_x, \Pi_y] = -ie\hbar B = -i\frac{\hbar^2}{l_B^2} \tag{1.1.4}$$

where we have introduced the *magnetic length* $l_B = \sqrt{\frac{\hbar}{eB}}$. This quantity is gauge invariant as expected and we can also see that the components of gauge invariant momentum are mutually conjugate so only one can be diagonalized at the time. Now we want to diagonalize the Hamiltonian and find the energies of a system. This can be done by introducing the ladder operators,[3] similar as in a treatment of a simple harmonic oscillator:

$$a = \frac{1}{\sqrt{2}}(\frac{x}{x_0} - i\frac{p}{p_0})$$

and $a^\dagger$ that we get by complex conjugation of a. Here we have introduced some normalization constants $x_0 = \sqrt{\frac{\hbar}{m\omega}}$ and $p_0 = \sqrt{\hbar m\omega}$, where ω is oscillator frequency. We can check the commutation relation for a and $a^\dagger$ to see that $[a, a^\dagger] = 1$ as it should be. In the case of a particle in magnetic field the ladder operators will be:

$$a = \frac{l_B}{\sqrt{2}\hbar}(\Pi_x - i\Pi_y) \tag{1.1.5}$$

and similarly $a^\dagger$. These relations can be inverted to express Π in terms of ladder operators:

$$\begin{aligned} \Pi_x &= \frac{\hbar}{\sqrt{2}l_B}(a + a^\dagger) \\ \Pi_y &= \frac{\hbar}{\sqrt{2}l_B}(a^\dagger - a) \end{aligned} \tag{1.1.6}$$

These new relations can help us diagonalize the Hamiltonian. The ladder operators obey similar rules as in a case of the harmonic oscillator when they act on a state:

$$\begin{aligned} a\,|n\rangle &= \sqrt{n}\,|n-1\rangle \\ a^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle \end{aligned} \tag{1.1.7}$$

If a acts on a ground states it will produce zero $a\,|0\rangle = 0$. We can build every other state by an application of $a^\dagger$ on a ground state:

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle$$

1.1.2 Landau levels

Let us now rewrite the Hamiltonian in terms of newly introduced ladder operators:

$$\hat{H}_B = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2) = \frac{\hbar^2}{ml_B^2}(a^\dagger a + \frac{1}{2}) \tag{1.1.8}$$

or if we introduce the cyclotron frequency $\omega_c = \frac{\hbar}{ml_B^2}$ we get:

$$\hat{H} = \hbar\omega_c(a^\dagger a + \frac{1}{2}) \tag{1.1.9}$$

Solving the Schrodinger equation

$$\hat{H}\,|n\rangle = E_n\,|n\rangle \tag{1.1.10}$$

so using the rules from the last section, we get that the energies of the system are:

$$E_n = \hbar\omega_c(n + \frac{1}{2}) \tag{1.1.11}$$

We see that the energy levels are labeled by the index n and these levels are called *Landau levels*(**LL**). One thing that can be observed immediately is that particle is characterized with only one pair of conjugate operators but in (1.1.2) we can see that two pairs of conjugate operators describe the Hamiltonian. This means that there is another degree of freedom that will make Landau levels degenerate.

1.1.3 Wave function

Before we tackle the issue of LL degeneracy first, we need to find the wave functions. For this purpose we will use Landau gauge $\mathbf{A} = B(-y, 0, 0)$. In this gauge the Hamiltonian is:

$$\hat{H}_B = \frac{(p_x - eBy)^2}{2m} + \frac{p_y^2}{2m} \tag{1.1.12}$$

We can see that $[H, p_x] = 0$ so the wave function can be put in a form $\Psi_{n,k} = \frac{e^{ikx}}{\sqrt{L}} f_{n,k}(y)$. If we let Hamiltonian act on this form of wave function we would be left with a Hamiltonian:

$$\hat{H}_B = \frac{p_y^2}{2m} + \frac{1}{2}m\omega_c(y - y_0)^2 \tag{1.1.13}$$

where $y_0 = kl_B^2$. This is precisely the harmonic oscillator Hamiltonian with the displacement of oscillation centers by y_0. The eigenfunctions are Hermite polynomials:

$$f_{n,k}(y) = H_n\left(\frac{y-y_0}{l_B}\right) e^{-\frac{(y-y_0)^2}{4l_B^2}} \tag{1.1.14}$$

The final wave function has the form:

$$\Psi_{n,k} = \frac{e^{ikx}}{\sqrt{L}} H_n\left(\frac{y-y_0}{l_B}\right) e^{-\frac{(y-y_0)^2}{4l_B^2}} \tag{1.1.15}$$

From this, we can see that in this particular gauge it looks like the function is smeared over x coordinate while localized around y_0 in y direction.

1.1.4 Degeneracy of Landau levels

Because of freedom in x direction, we will impose the periodic boundary conditions so we can count the number of states. Boundary conditions lead to a quantization of wave vector $k = i\frac{2\pi}{L}$, where $i = 1, 2, 3 \ldots M$. The gauge we have chosen is suitable for rectangular geometries so let say that in y direction sample has a width D. Two states are separated by distance $\Delta y = \Delta k l_B^2$ and using the quantization relation of wave vector $\Delta y = \frac{2\pi l_B^2}{L}$, so the one state occupies the area of $S = \Delta y L = 2\pi l_B^2$. The total number of possible states in one LL is, therefore:

$$N_B = \frac{LD}{S} = \frac{LD}{2\pi l_B^2} = n_B LD \tag{1.1.16}$$

where $n_B = \frac{1}{2\pi l_B^2} = \frac{eB}{h}$ is flux density which represents the magnetic field expressed in flux quantum $\frac{h}{e}$.

So the degeneracy of LL is equal to the number of flux quanta piercing the surface area.

Another useful quantity to define is the ratio between the number of electrons and the flux quanta, **filling factor**:

$$\nu = \frac{N_{el}}{N_B} = \frac{n_{el}}{n_B} = \frac{h n_{el}}{eB} \tag{1.1.17}$$

1.1.5 Symmetric gauge

In the previous sections we have derived the eigenfunction in Landau gauge, and here we will do the same procedure for the symmetric gauge as it will be more useful in the later discussions of Laughlin wave function and FQHE. The symmetric gauge is $\mathbf{A} = \frac{B}{2}(-y, x, 0)$. In our discussion about the particle in magnetic filed we have introduced the gauge invariant momentum, now in

order to find the eigenfunctions in symmetric gauge we will introduce the new "momentum".

$$\tilde{\mathbf{\Pi}} = \mathbf{p} - e\mathbf{A} \tag{1.1.18}$$

as it can be seen this quantity only differs from the previous momentum by the sign, which makes it gauge variant and therefore not a physical quantity, so the special care needs to be taken in dealing with it. The commutation relation between different components is different up to the sign from the previous:

$$[\tilde{\Pi}_x, \tilde{\Pi}_y] = ie\hbar B \tag{1.1.19}$$

The sad thing is that the new momentum does not commute with the old one and cannot be diagonalized at the same time:

$$\begin{aligned} [\Pi_x, \tilde{\Pi}_x] &= 2ie\hbar \frac{\partial A_x}{\partial x} \\ [\Pi_y, \tilde{\Pi}_y] &= 2ie\hbar \frac{\partial A_y}{\partial y} \\ [\Pi_x, \tilde{\Pi}_y] &= [\Pi_y, \tilde{\Pi}_x] = ie\hbar \left(\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x} \right) \end{aligned} \tag{1.1.20}$$

However, the lucky coincidence is that in the symmetric gauge they do commute. We can now define the new set of ladder operators:

$$\begin{aligned} b &= \frac{l_B}{\sqrt{2}\hbar}(\tilde{\Pi}_x + i\tilde{\Pi}_y) \\ b^\dagger &= \frac{l_B}{\sqrt{2}\hbar}(\tilde{\Pi}_x - i\tilde{\Pi}_y) \end{aligned}$$

As discussed earlier the Landau levels are highly degenerate and beside the quantum number n that labels the different levels, we need to have another which will distinguish between the different states inside the same Landau level. The complete state would then be characterized by two quantum numbers, n and m:

$$|n, m\rangle = |n\rangle \otimes |m\rangle \tag{1.1.21}$$

These new operators b and $b^\dagger$ will precisely enumerate the degeneracy inside the Landau levels:

$$\begin{aligned} b\,|n, m\rangle &= \sqrt{m}\,|n, m-1\rangle \\ b^\dagger\,|n, m\rangle &= \sqrt{m+1}\,|n, m+1\rangle \end{aligned} \tag{1.1.22}$$

Now we can create any state with the application of right ladder operators:

$$|n, m\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} \frac{(b^\dagger)^m}{\sqrt{m!}} |0, 0\rangle \tag{1.1.23}$$

We are only interested in the LLL ($n = 0$) for which applies

$$a\,|0, m\rangle = 0 \tag{1.1.24}$$

In order to solve this, we need to convert it to the differential equation. Using (1.1.5) and momentum operator in coordinate representation we can express the a operator:

$$a = -i\sqrt{2}\left(\frac{l_B}{2}(\partial_x - i\partial_y) + \frac{x + iy}{4l_B}\right) \tag{1.1.25}$$

This can be put in more convenient form if we introduce the complex coordinates. We can define $z = x + iy$, $\partial = \frac{1}{2}(\partial_x + i\partial_y)$ and $\bar{\partial} = \frac{1}{2}(\partial_x - i\partial_y)$. Now the differential equation reads:

$$\left(\frac{z}{4l_B} + l_B\bar{\partial}\right)\psi_{0,m}(z, z^*) = 0 \tag{1.1.26}$$

This can be solved by

$$\psi_{0,m}(z, z^*) = f(z)e^{-\frac{|z|^2}{4l_B^2}} \tag{1.1.27}$$

where $f(z)$ is analytic function because $\bar{\partial}f(z) = 0$. From here we can see that there is another degree of freedom. We can choose any analytic function for $f(z)$, and that is connected to the second quantum number m mentioned earlier. To further determine the wave function we need to express the b operators the same as we did for a:

$$b = -i\sqrt{2}\left(\frac{z^*}{4l_B} + l_B\partial\right) \tag{1.1.28}$$

Same as we did with the a and LLL we can do with b and $m = 0$ to yield the equation:

$$(z^* + 4l_B^2\partial)\psi'_{n,0}(z, z^*) = 0 \tag{1.1.29}$$

This equation can be solved by:

$$\psi'_{n,0}(z, z^*) = g(z^*)e^{-\frac{|z|^2}{4l_B^2}} \tag{1.1.30}$$

where $g(z^*)$ is the anti-analytic $\partial g(z^*) = 0$. So the wave function $\psi_{0,0}(z, z^*)$ must have a prefactor that is analytic and anti-analytic at the same time, it is a constant fixed by the normalization condition:

$$\psi_{0,0}(z, z^*) = \frac{1}{\sqrt{2\pi l_B^2}}e^{-\frac{|z|^2}{4l_B^2}} \tag{1.1.31}$$

From this we can construct a state inside LLL with an arbitrary m by acting with $b^\dagger$ on the $\psi_{0,0}(z, z^*)$. So we get:

$$\psi_{0,m}(z, z^*) = \frac{i^m}{\sqrt{2\pi l_B^2 m!}}\left(\frac{z}{\sqrt{2}l_B}\right)^m e^{-\frac{|z|^2}{4l_B^2}} \tag{1.1.32}$$

alternatively said the LLL is spanned by the Gaussian multiplied by polynomial basis z^m, $\psi_{0,m} \sim z^m e^{-\frac{|z|^2}{4}}$, where we redefined $z = \frac{x+iy}{l_B}$ and left out the normalization factor.

We can also verify that in this gauge we get the same number for the degeneracy of LL. This wave function is peaked at the radius $r = \sqrt{2m} l_B$, for some maximal m_{max} the area of state would be $S = 2m_{max} l_B^2 \pi$ so the number of states in sample area A is $m_{max} = \frac{A}{2\pi l_B^2} = n_B * A = N_B$ same as before, equal to the number of flux quanta threading the system.

1.2 Wannier functions

Usually, when considering periodic systems and independent particle approximation, we choose Bloch functions for the basis. However, these functions are not the only solutions, for example, we can choose more localized Wannier functions.[4] Wannier functions can be seen as an LMO (localized molecular orbitals) for solid-state physics.

1.2.1 Bloch and Wannier functions

For a periodic system, the one-particle effective Hamiltonian $\hat{H}$ commutes with lattice-translation operator $T_{\mathbf{R}}$ so that the eigenstate can be:

$$\psi_{n,k} = u_{n,k}(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{1.2.1}$$

where $u_{n,k}(\mathbf{r})$ is an arbitrary function with a periodicity of the Hamiltonian. Using Bloch functions and superposing functions with different $\mathbf{k}$ we can build a localized packet in real space. By Heisenberg uncertainty principle the broader the range of $\mathbf{k}$ the more localization in real space. However, as we said at the beginning, we are considering a periodic system, so the $\mathbf{k}$ lives in a periodic Brillouin zone (BZ). We can now construct a Wannier function as an integral over the first BZ:

$$\omega_0(\mathbf{r}) = \frac{V}{(2\pi)^3} \int_{BZ} \mathrm{d}\mathbf{k} \psi_{n,k}(\mathbf{r}) \tag{1.2.2}$$

where V is real-space primitive cell volume. This can be generalized if we apply real space translation operator $e^{-i\mathbf{k}\cdot\mathbf{R}}$ to the $\omega_0(\mathbf{r})$ in order to generate a function at the position $\mathbf{R}$ (real space lattice vector). The general form would be:

$$\omega_{n,\mathbf{R}}(\mathbf{r}) = \frac{V}{(2\pi)^3} \int_{BZ} \mathrm{d}\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{R}} \psi_{n,k}(\mathbf{r}) \tag{1.2.3}$$

or in the more compact Dirac notation

$$|\mathbf{R}, n\rangle = \frac{V}{(2\pi)^3} \int_{BZ} \mathrm{d}\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{R}} |\psi_{n,k}\rangle \tag{1.2.4}$$

These functions $|\mathbf{R}, n\rangle$ form an orthonormal set which makes them perfect for basis use. The inverse transformation, if we want to go from Wannier function to Bloch states is:

$$|\psi_{n,k}\rangle = \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} |\mathbf{R}, n\rangle \tag{1.2.5}$$

This means that as the bunch of spread Bloch functions can be used to make a localized wave-packet, in the same way, the superposition of localized wave-packets can be used to build up a spread Bloch state. What one should bear in mind is that by gaining the localization of states we lose the eigenfunction property of the Hamiltonian, as these functions are not the eigenstates of the original Hamiltonian.

1.3 Hall effect

Let us look at the system in figure below

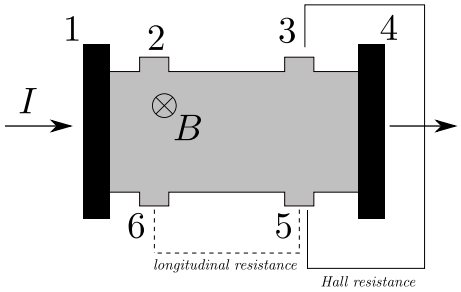


Figure 1.1: Hall system

It is a 2D system of electrons in a strong magnetic field perpendicular to the plane of motion. Let us say that contacts 1 and 4 are on different potentials and that current flows through them. With this kind of setup, we can measure the longitudinal and transverse resistance. These two quantities can be easily differentiated topologically. Draw a line from contacts 1 to 4 where the current flows, now if the line connecting two contacts crosses this imaginary current line, the resistance measured that way would be transverse, or longitudinal otherwise. Transverse resistance is often called Hall resistance R_H.

1.3.1 Classical Hall effect

Edwin Hall showed that transverse resistance of thin metallic plate subjected to the perpendicular magnetic field $\mathbf{B}$ would be:

$$R_H = \frac{B}{qn_{el}}$$

where the sign of the q depends on the nature of current carriers, and n_{el} is the electron density in the sample. The simple explanation would be that this effect is a consequence of Lorentz force which affects the electron trajectory in the sample. The slightly better explanation can be deduced from the Drude model of diffusive transport in metals. This model has several assumptions:

- Metals is formed from a sea of positively charged ions and detached electrons
- The long range interactions between electrons are neglected. Only interaction the electron has with the rest of the system is through instant collisions, separated on average by time τ
- After the collision the velocity of the electron only depends on the local temperature distribution and is independent of the velocity before the collision

These assumptions can be expressed as the equation:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -e\left(\mathbf{E} + \frac{\mathbf{p}}{m} \times \mathbf{B}\right) - \frac{\mathbf{p}}{\tau} \tag{1.3.1}$$

we are assuming that the current carriers are electrons. We are looking for a stationary solution so $\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = 0$, the magnetic field is perpendicular to the plane of motion $\mathbf{B} = B\hat{e}_z$:

$$\begin{aligned} eE_x &= -\frac{eB}{m}p_y - \frac{p_x}{\tau} \\ eE_y &= \frac{eB}{m}p_x - \frac{p_y}{\tau} \end{aligned} \tag{1.3.2}$$

We can rewrite these equations by introducing the Drude conductivity $\sigma_0 = \frac{n_{el}e^2\tau}{m}$ and cyclotron frequency:

$$\sigma_0 E_x = -en_{el}\frac{p_x}{m} - en_{el}\frac{p_y}{m}(\omega_c\tau) \tag{1.3.3}$$

$$\sigma_0 E_y = en_{el}\frac{p_x}{m}(\omega_c\tau) - en_{el}\frac{p_y}{m} \tag{1.3.4}$$

If we further introduce the current density:

$$\mathbf{j} = -en_{el}\frac{\mathbf{p}}{m} \tag{1.3.5}$$

in the matrix form of the Ohm law $\mathbf{E} = \rho\mathbf{j}$ we have for the resistivity tensor:

$$\rho = \sigma^{-1} = \sigma_0^{-1}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} \tag{1.3.6}$$

From the resistivity tensor we can read the transverse (Hall) resistivity as

$$\rho_H = \frac{\omega_c \tau}{\sigma_0} = \frac{eB\tau}{m} \frac{m}{n_{el} e^2 \tau} = \frac{B}{e n_{el}} \tag{1.3.7}$$

The conductivity tensor could be found as an inverse of resistivity, and after a simple algebra we will get:

$$\sigma = \begin{pmatrix} \frac{\sigma_0}{(1+\omega_c^2\tau^2)} & -\frac{\sigma_0 \omega_c \tau}{(1+\omega_c^2\tau^2)} \\ \frac{\sigma_0 \omega_c \tau}{(1+\omega_c^2\tau^2)} & \frac{\sigma_0}{(1+\omega_c^2\tau^2)} \end{pmatrix} \tag{1.3.8}$$

It is interesting to notice that in the limit of a clean sample $\omega_c \tau \to \infty$ (without the impurity for electrons to bounce off) we see that the off-diagonal elements entirely govern both resistivity and conductivity.

One may observe that at the beginning we were talking about the resistance, but our calculations with the Drude model led us to the resistivity. Resistance is the property of the sample and depends on the geometry and material that sample is made of. Resistivity, on the other hand, is a property of the material itself. These two quantities are related by $R = \rho \frac{L}{A}$, where L is the length of the conductor, and A is its cross-section area. From a dimensional point of view, the cross-section area of a d-dimensional conductor scales as L^{d-1}. So the resistance will scale as $R \sim \rho L^{d-2}$ which means that for 2D conductor the resistance and resistivity are the same from the dimensional point of view. This argument assumes a square sample which is not generally true. In the general case, the resistance would depend on the ration $\frac{L}{W}$ or some factor of it. However, in the case of Hall resistance, the length of the sample itself is the cross-section, so the factor is equal to one.

1.3.2 Quantum Hall effect

By the end of the twentieth century, a more interesting and fascinating version of this effect was experimentally discovered. The first one, integer quantum Hall effect (IQHE), was observed in 1980. by v. Klitzing, Dorda, and Pepper for which v. Klitzing won the Nobel price in 1985.

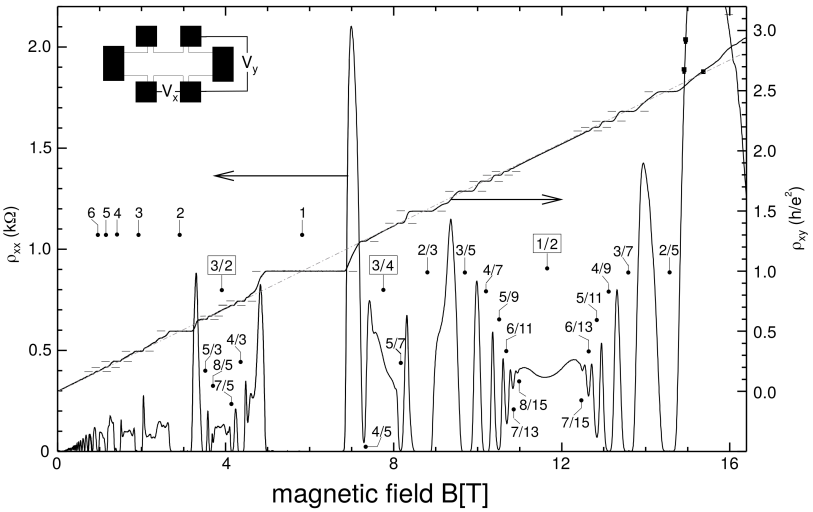


Figure 1.2: Experimental data for Hall and longitudinal resistivity[5]

They have discovered that Hall resistance exhibits strange plateau-like behavior and deviates from the linear dependence on B. These places where Hall resistance would show plateau are precisely quantized like

$$R_H = \frac{1}{n}\frac{h}{e^2} \tag{1.3.9}$$

where $n \in \mathbb{N}$. These plateaus were followed by and complete vanishing of longitudinal resistance. An interesting property is that this quantization does not depend on the properties of a material, but it is universal. The vanishing of longitudinal resistance indicates a connection with the limit of the impurity-free Drude model and suggests the universality of the effect independent on the impurities in the sample. The even more interesting effect was discovered three years after the IQHE by Tsui, Störmer and Gossard, fractional quantum Hall effect (FQHE) for $n = \frac{1}{3}$. These two effects, even if exhibit the similar kind of behavior, have very different physical explanations behind them. We will not further discuss the IQHE but will focus our attention at FQHE. The origin of FQHE is in the strong electronic correlations and Coulomb interaction between the electrons. Laughlin showed[6] that the origin of the observed FQHE at $\frac{1}{3}$ is

due to the formation of correlated incompressible electron liquid. The incompressible means that if we vary the filling factor, we will not pay the infinitesimal amount of energy. This can be understood for the system with the fixed number of particles, if we decrease the magnetic field then automatically the surface area occupied by one state will shrink while the total area of the system remains the same. An infinitesimal change in the energy cannot accommodate this increase of the state surface because of the finite gap between LL.

1.3.3 Fractional quantum Hall effect (FQHE)

As derived earlier the wave function for LLL is:

$$\psi_m \sim z^m e^{-\frac{|z|^2}{4}} \tag{1.3.10}$$

If we are restricted to the LLL we can see that only one wave function with a given m exists, and because of the analyticity $m \geq 0$. Before we tackle the issue of many particles let us solve the two particles problem. Consider arbitrary central potential $V(|\mathbf{r}_1 - \mathbf{r}_2|)$. We could try and follow the standard procedure of writing the relative momentum part and using the conservation of angular momentum and then solving for the radial part, or we can use the analytic properties of the function. If we already have the angular part we automatically have the radial from the analytic properties of LLL wave functions. We can use this to write down the two body wave function with relative angular momentum m and center of mass angular momentum M:

$$\Psi_{m,M}(z_1, z_2) \sim (z_1 + z_2)^M (z_1 - z_2)^m e^{-\frac{|z_1|^2+|z_2|^2}{4}} \tag{1.3.11}$$

If we are dealing with spinless fermions, m has to be odd because of the symmetric properties and if $M \geq 0$ than we have just a Gaussian prefactor that is nothing else than a linear combination of single-particle wave functions in the LLL. What is remarkable about this function is (if we neglect LL mixing) that this is the exact wave function for every central potential in LLL.

Laughlin wave function is a generalization of two body wave function for N particles:[7,6]

$$\psi_m^L(z, z^*) = \prod_{k<l} (z_k - z_l)^m e^{\sum_j \frac{|z_j|^2}{4}} \tag{1.3.12}$$

Immediately we can notice that there is no center of mass dependence, that is because that would destroy the spatial symmetry and introduce the Goldstone bosons and such system would be gapless.

Since this is the wave function of electrons the exponent m needs to be of the form $m = 2k+1$ in order to satisfy the symmetry condition. We are going to

show that the filling factor fixes this exponent m. From the Laughlin function if we fix one particle, e.g., z_1 we can see that the highest power is $z_1^{m(N-1)}$ and from the earlier discussion in 1.1.5 we saw that the power of z is fixed by the N_B, or $mN = N_B$ for $N \gg 1$, or

$$m = \frac{N_B}{N} = \frac{1}{\nu} \tag{1.3.13}$$

But because of the fermionic nature we know:

$$\nu = \frac{1}{m} = \frac{1}{2k+1} \tag{1.3.14}$$

or that Laughlin wave function is a candidate for filling factors $1, \frac{1}{3}, \frac{1}{5}, \ldots$

1.3.4 Fully filled non-interacting LL $\nu = 1$

From the general form of the Laughlin wave function, it looks that it should also describe $\nu = 1$ case of fully filed non-interacting LL. We can check that because we know how to build these functions, we should simply anti-symmetries over all particles. Let us start with a simple situation and then generalize. In general LL wave function has the form:[7]

$$\psi([z]) = f([z])e^{-\frac{1}{4}\sum_j |z_j|^2} \tag{1.3.15}$$

where $[z]$ represents all the particles. Consider now the case for two particles:

$$f([z]) = \begin{vmatrix} (z_1)^0 & (z_2)^0 \\ (z_1)^1 & (z_2)^1 \end{vmatrix} = (z_1)^0(z_2)^1 - (z_2)^0(z_1)^1 = z_1 - z_2 \tag{1.3.16}$$

as this is the lowest order polynomial that is asymmetric. For the three particles we have:

$$\begin{aligned} f([z]) &= \begin{vmatrix} (z_1)^0 & (z_2)^0 & (z_3)^0 \\ (z_1)^1 & (z_2)^1 & (z_3)^1 \\ (z_1)^2 & (z_2)^2 & (z_3)^2 \end{vmatrix} = z_2z_3^2 - z_3z_2^2 - z_1z_3^2 + z_3z_1^2 + z_1z_2^2 - z_2z_1^2 \\ &= -(z_1 - z_2)(z_1 - z_3)(z_2 - z_3) \\ &= -\prod_{i<j}^{3}(z_i - z_j) \end{aligned} \tag{1.3.17}$$

This can now easily be generalized for N particles:

$$f_N([z]) = \prod_{i<j}^{N}(z_i - z_j) \tag{1.3.18}$$

which is the same as Laughlin wave function for $m = 1$. So far the Laughlin proposal showed some interesting properties, but we did conclude that it truly represents the ground state of FQHE. In the next several sections we will discuss the energy properties and excitation states of this function.

1.3.5 Haldane's pseudo-potentials

Before we dive into energy characteristics of Laughlin state, let us return to the two body case. As stated before two body wave function is the exact wave function for every central potential in LLL. We can decompose the given potential into:[7,3]

$$v_m = \frac{\langle m, M|V|m, M\rangle}{\langle m, M|m, M\rangle} \tag{1.3.19}$$

The reason v does not depend on M is that we are considering a central potential. The values v_m obtained this way are called **Haldane's pseudo-potentials**. These energies determine the spectrum of two body state entirely because the magnetic field quenches the kinetic energy and particles are trapped inside the single LL. As we said earlier, that peak value of this wave function is around $r \sim \sqrt{2m}l_B$, that means Haldane's pseudo-potentials represent the approximate value of the original potential at that radius:

$$v_m \sim V(|z| = \sqrt{2m}l_B) \tag{1.3.20}$$

We have introduced these quantities because they can be useful in describing the N particle spectrum. The general N particle interaction potential can be decomposed as:

$$V = \sum_{i<j} V(|z_i - z_j|) = \sum_{i<j} \sum_{m'=0}^{\infty} v_{m'} P_{m'}(i, j) \tag{1.3.21}$$

where $P_{m'}(i, j)$ is a projector of particle pair (i, j) to the subspace of a relative angular momentum m'. If we now return to the Laughlin wave function, we can see that from the polynomial part no particle is in the state $m' < m$.

We can choose a model potential to be:

$$v'_m = \begin{cases} 1 & \text{for } m < m' \\ 0 & \text{for } m \geq m' \end{cases} \tag{1.3.22}$$

From this, we can see that Laughlin wave function is the exact eigenstate with zero energy, $V\psi^L_m = 0$. Because the model is entirely repulsive (all energies must be $E \geq 0$) this is also the exact ground state. We can see from this that even the interaction is repulsive we have a discrete energy spectrum. This can be explained by the fact that kinetic energy to which would the excessive energy go if two particles would approach each other is now quenched by the magnetic field and frozen to the one LL by the energy gap. Laughlin state is also the only zero energy state.

We can easily see this if we keep the number of particles and flux fixed. Any excited state has a different value of angular momentum from m. If the state

has angular momentum m' such that $m' < m$ automatically follows that it costs energy. If on the other hand we have a state with angular momentum $m' > m$, the mean distance between the electrons will grow but because of the fixed sample surface and filling factor, at least one pair of electrons will be in a state with angular momentum $m' < m$ and that will cost energy. This means that any excitation from the Laughlin state will require the finite amount of energy $v_{m'}$, in other words, the Laughlin's wave function represents an incompressible state.

The model given above may seem too artificial and far from the reality, but if we numerically compare the wave function from this model and one from Coulomb potential decomposed in Haldane's pseudo-potentials we would get an overlap of more than 99%. This high level of overlap may be understood in a way that decomposition of Coulomb potential in Haldane's pseudo-potentials is monotonically decreasing function. The value of v_1 is much larger than v_3 which means that all the subsequent terms can be treated by the perturbative approach and the ground state is protected.

1.3.6 Laughlin state excitations

So far only the ground state properties of the Laughlin function were discussed, but we also need to examine the excitations. There are two types of excitations, collective neutral and charged. Collective neutral excitations exhibit similar kind of phonon behavior as in a superfluid or solids.[7] They are a charge-density-wave excitations with magneto-roton minimum that is gapped at $\mathbf{k} = 0$. We will focus our attention at the charged excitations.

1.3.7 Charged excitations

Quasi-hole excitations

If we recall, the changes in filling factor will introduce excitations in our system. From the definition of filling factor, this can be achieved in two ways, by changing the number of particles or changing the number of flux quanta. We know that the number of flux is related to the number of zeros in Laughlin function so that we can propose a function for an excited state:

$$\psi^{\mathrm{qh}}_{L,m} = \prod_{j=1}^{N}(z_j - z_0)\psi^{L}_{m}([z]) \tag{1.3.23}$$

if we imagine the symmetric gauge and assume $z_0 = (0, 0)$ this means that we have added an another zero at the center of the system. Let us now verify that

proposed function is indeed an excited state. We can do that by expanding it in a polynomial:

$$\psi^{\text{qh}}_{L,m} = \sum_{m_i} \gamma_{m_1,m_2,\ldots,m_N} z_1^{m_1+1} z_2^{m_2+1} \ldots z_N^{m_N+1} e^{-\frac{1}{4}\sum_j |z_j|^2} \tag{1.3.24}$$

where $\gamma_{m_1,m_2,\ldots,m_N}$ is an expansion coefficient. We can see that each of the exponents was increased $m_i \to m_i + 1$. This physically means that by adding an excitation at the center of the system we have forced each electron to jump to the next angular momentum state, and we are left with a state $m = 0$ empty. Because of the empty state, this is a quasi-hole excitation. We saw that changing the number of flux $N_B \to N_B + 1$ we have introduced a quasi-hole excitation and lowered the filling by a small amount. However, if we keep the filling factor fixed and change only the flux, we would need to introduce some charge to compensate for this change. From the definition of filling factor, we have $m\Delta N = \Delta N_B$ which means:

$$\Delta N = \frac{\Delta N_B}{m} \tag{1.3.25}$$

If we introduce the change of flux by 1, in order to compensate that we would need to add a **fractional charge** of $\frac{e}{m}$, so the quasi-hole has a fractional charge. From the wave function, it follows that particles gain an additional phase of 2π by making one circulation around the center of the system compared to the original situation (vorticity):

$$\psi^{\text{qh}}_{L,m} \propto \prod_j^N e^{-i\theta_j} \psi^L_m \tag{1.3.26}$$

This behavior is similar to the vortex type behavior in type-II superconductors.

Quasi-particle excitations

As we saw in the previous section one may wonder if there is a way to lower the number of flux quanta and create quasi-particle excitations. Naturally, these excitations should have opposite vorticity, so the first suggestions for the wave function prefactor would be $\prod_j^N (z_j^* - z_0^*)$. However, there is a problem with this, by introducing the complex conjugate we are breaking the condition of analicity and introducing LL mixing. We can fix this problem by projecting this function to the LLL:

$$\psi^{\text{qp}}_{L,m} = \mathcal{P}_{LLL} \prod_j^N (z_j^* - z_0^*) \psi^L_m([z]) \tag{1.3.27}$$

1.3.8 LLL projection

Here we will examine in more details the projection into LLL.[8] We have seen the general form of LL wave functions (1.3.15), and we know that in the LLL $f([z])$ need to be analytical. Let us consider the set Θ of entire functions of N complex variables. As an example for $N = 1$, z^2 is one such function but z^* is not. Now we will define the inner product on Θ:

$$(f, g) = \int \mathrm{d}\xi[z] f^*[z] g[z] \tag{1.3.28}$$

where the measure is defined like:

$$\mathrm{d}\xi[z] = \prod_{i=1}^{N} \frac{1}{2\pi} e^{-\frac{1}{2}|z_i|^2} \mathrm{d}x_i \mathrm{d}y_y \tag{1.3.29}$$

also, we are only considering the functions with a finite norm $(f, f) < \infty$. By defining the norm (1.3.29) we now have the inner product of two wave functions to be:

$$\left\langle \psi_m^{\prime L} \middle| \psi_m^L \right\rangle = (f', f) \tag{1.3.30}$$

We can now work with f's which are analytical instead of ψ which in general are not. Because of the analytical condition on f's the number of allowed operations is very small. There are only three types that satisfy this condition, (1) multiplication by a constant, (2) multiplication by a power of z and (3) differentiation with respect to z. To analyze these operators, we will introduce the orthonormal basis:

$$f_n(z) = \frac{z^n}{\sqrt{2^n n!}} \tag{1.3.31}$$

This basis has a certain properties:

$$\begin{aligned} z f_m &= \sqrt{2(m+1)} f_{m+1} \\ \frac{\mathrm{d}f_m}{\mathrm{d}z} &= \sqrt{\frac{m}{2}} f_{m-1} \end{aligned} \tag{1.3.32}$$

from these, we see that we can introduce a ladder like operators that will move between different f_m

$$\begin{aligned} a^\dagger &= \frac{z}{\sqrt{2}} \\ a &= \sqrt{2} \frac{\mathrm{d}}{\mathrm{d}z} \end{aligned} \tag{1.3.33}$$

In the previous section, we have encountered a projector to the LLL. This was necessary in order to fit the function z^* into LLL. For example consider an arbitrary matrix element $(f_n, z_k^* f_m)$, we know that $z^* f_m$ is outside the Hilbert

space, but the matrix element is well defined. From the definition of the inner product we have:

$$(f_n, z_k^* f_m) = (z_k f_n, f_m) \tag{1.3.34}$$

Combining the two equations from (1.3.33) we see that:

$$z_k^* = 2\frac{\partial}{\partial z_k} \tag{1.3.35}$$

Now we have obtained the projection of z^* to the LLL. These two operators are not quite the same, as it can be seen they do not commute the same way with z_k. For z_k^* we have:

$$(f, z_k z_k^* g) = (f, z_k^* z_k g) \tag{1.3.36}$$

but for the $\frac{\partial}{\partial z_k}$ we have:

$$\left(f, z_k 2\frac{\partial}{\partial z_k}, g\right) \neq \left(f, 2\frac{\partial}{\partial z_k} z_k, g\right) \tag{1.3.37}$$

so we need to differentiate between the Hermitian conjugate and the adjoint.

Chapter 2

Magnetic Wannier functions

2.1 Motivation

In the previous literature[9] a similar problem was studied. Rashba et al. were considering magnetic Wannier functions in a rationally symmetric gauge. A Gaussian wave function was created on a lattice point and afterwards by the magnetic translation operator moved to other lattice points. Because of the non-trivial Chern number of the problem, no complete localization is possible. The fastest way for localization would be an envelope function with a Thouless critical exponent of r^{-2}.

In an another paper[10] by Panfilov et al. an effective lattice Hamiltonian was constructed using the magnetic Wannier functions of Rashba et al. The final Hamiltonian has the strong on site repulsion and a chiral interaction, reminiscent of a magnetic field presence. The amplitude of the chiral interactions depends on the nearest neighbor site densities. The chiral interaction explicitly breaks the time reversal symmetry. The amplitudes for matrix elements contained Bessel functions and as such might be a problem for an experimental realization. Our goal is to circumvent these problems by working in a Landau gauge, naturally asymmetric, and try to extract wire-like construction.[1] The effective Hamiltonian should be without a complicated amplitude expressions and simple for an experimental realization.

2.2 Introduction

We are considering a LLL in square geometry on a torus $\mathbb{T} = [0, Lx] \times [0, Ly] \sim [0, N_x \mathbf{e_x}] \times [0, N_y \mathbf{e_y}]$. Magnetic field is perpendicular to the system plane $\mathbf{B} = B\mathbf{e_z}$, and we are working in a Landau gauge $\mathbf{A} = (0, Bx, 0)$. The number of flux quanta threading the system is $N_\Phi = \frac{L_x L_y}{2\pi l_B^2} = N_x N_y$. Model Hamiltonian for which the Laughlin bosonic state is an exact zero energy state is:

$$\hat{H} = \int \mathrm{d}\mathbf{x_1} \int \mathrm{d}\mathbf{x_2} V \delta(\mathbf{x_1} - \mathbf{x_2}) \psi^\dagger(\mathbf{x_1}) \psi^\dagger(\mathbf{x_2}) \psi(\mathbf{x_2}) \psi(\mathbf{x_1}) \tag{2.2.1}$$

where $\psi(\mathbf{x}) = \sum_n \phi_n^\star(\mathbf{x}) a_n$ and $\phi_n(\mathbf{x}), n = 1, \ldots, N_\Phi$ an arbitrary basis in LLL in the coordinate representation. We take $\phi_n(\mathbf{x})$ to be magnetic Bloch states of Wu et al.[2]

$$|\phi_{\mathbf{k}}\rangle \equiv |\mathbf{k}\rangle = \frac{1}{\sqrt{N_x}} \sum_{m=-[\frac{N_x}{2}]}^{[\frac{N_x}{2}]} e^{i2\pi m k_x / N_x} |j = mN_y + k_y\rangle \tag{2.2.2}$$

where $|j\rangle$ is the usual basis in a LLL on a torus:

$$\langle x, y | j \rangle = \frac{1}{(\sqrt{\pi} L_y l_B)^{1/2}} \sum_n^{\mathbb{Z}} \exp\left(2\pi (j + nN_\Phi) \frac{x + iy}{L_y} - \frac{\pi L_x}{N_\Phi L_y} (j + nN_\Phi)^2 \right) e^{-\frac{x^2}{2l_B^2}} \tag{2.2.3}$$

Bloch states $|\mathbf{k}\rangle$ have a quasi-periodicity $|k_x + N_x, k_y + N_y\rangle = e^{-\frac{2\pi i k_x}{N_x}} |k_x, k_y\rangle$. Because of δ function in model Hamiltonian, we can take for two-body interaction part, i.e., Hamiltonian:

$$\hat{H} = V \int \mathrm{d}\mathbf{q} \rho_{\mathbf{q}} \rho_{-\mathbf{q}} \tag{*}$$

where $\rho_{\mathbf{q}} = \int \mathrm{d}\mathbf{r} \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}}$. To describe the interaction in a magnetic Wannier basis i.e. over a lattice structure we take

$$\psi(\mathbf{r}) = \sum_{\mathbf{k}} \langle \mathbf{k} | \mathbf{r} \rangle \, a_{\mathbf{k}} \tag{2.2.4}$$

and

$$a_{\mathbf{k}} = \sum_{\mathbf{m}} e^{i\mathbf{k}\cdot\mathbf{m}} a_{\mathbf{m}} \tag{2.2.5}$$

where $\mathbf{m}(m_x, m_y)$ and $m_x = [-\frac{N_x}{2}, \frac{N_x}{2}]$ and $m_y = [-\frac{N_y}{2}, \frac{N_y}{2}]$.

2.3 Normalization of magnetic Bloch and Wannier states

First, we need to check the normalization of magnetic Bloch states:

$$\begin{aligned}\langle \mathbf{k}|\mathbf{k}'\rangle = &\frac{1}{N_x}\frac{1}{\sqrt{\pi}l_B}\sum_{m,m'=-[\frac{N_x}{2}]}^{[\frac{N_x}{2}]}\sum_{n,n'}^{\mathbb{Z}}\\ &\exp\left(\frac{2\pi i}{N_x}(m'k'_x - mk_x)\right)\\ &\exp\left(-\frac{\pi L_x}{N_\phi L_y}((m'N_y + k'_y + n'N_\phi)^2 + (mN_y + k_y + nN_\phi)^2)\right)\\ &\int_0^{L_x}\mathrm{d}x\exp\left(\frac{2\pi x}{L_y}(m'N_y + k'_y + n'N_\phi + mN_y + k_y + nN_\phi) - \frac{x^2}{l_B^2}\right)\\ &\delta_{m'N_y+k'_y+n'N_\phi,mN_y+k_y+nN_\phi}\end{aligned} \tag{2.3.1}$$

where we have used the definition of a δ function for the integration over y. This delta condition can be written in the form:

$$\begin{aligned}m'N_y + k'_y + n'N_\phi &= mN_y + k_y + nN_\phi\\ (m' - m + n'N_x - nN_x)N_y + (k'_y - k_y) &= 0\end{aligned}$$

Because k_y, k'_y are from the I Brillouin zone, in order for this equation to be true it must be $k'_y = k_y$, ie. $\delta_{k'_y,k_y}$. Now we need to examine the behavior of expression next to N_y. We have $m \in [0, N_x - 1]$ only when the equation will be true is $m = m'$ and $n = n'$, ie. finally we have the decomposition of a δ function:

$$\delta_{m'N_y+k'_y+n'N_\phi,mN_y+k_y+nN_\phi} = \delta_{k'_y,k_y}\delta_{m',m}\delta_{nN_\phi,n'N_\phi}$$

Now we would like to do the summation of primed values in (2.3.1).

$$\begin{aligned}\langle \mathbf{k}|\mathbf{k}'\rangle = &\frac{1}{\sqrt{\pi}l_B}\frac{1}{N_x}\sum_{m=-[\frac{N_x}{2}]}^{[\frac{N_x}{2}]}\exp\left(\frac{2\pi i m}{N_x}(k'_x - k_x)\right)\delta_{k'_y,k_y}\\ &\underbrace{\sum_n^{\mathbb{Z}}\int_0^{L_x}\mathrm{d}x\exp\left(-\frac{x^2}{l_B^2} + \frac{4\pi x}{L_y}(mN_y + k_y + nN_\phi) - \frac{4\pi^2 l_B^2}{L_y^2}(mN_y + k_y + nN_\phi)^2\right)}_{I}\end{aligned} \tag{2.3.2}$$

We can isolate I and solve it separately. The goal is to write it in the form of a Gaussian integral that in combination with an infinite sum will lead to

integration limits from $-\infty$ to ∞.

$$I = \sum_n^{\mathbb{Z}} \int_0^{L_x} \mathrm{d}x \exp\left(-\frac{x^2}{l_B^2} + \frac{4\pi x}{L_y}(mN_y + k_y + nN_\phi) - \frac{4\pi^2 l_B^2}{L_y^2}(mN_y + k_y + nN_\phi)^2\right)$$
$$= \sum_n^{\mathbb{Z}} \int_0^{L_x} \mathrm{d}x \exp\left(-(\frac{x}{l_B} - \frac{2\pi}{L_y}(mN_y + k_y + nN_\phi))^2\right) =$$
$$\sum_n^{\mathbb{Z}} \int_{-\frac{2\pi}{L_y}(mN_y+k_y+nN_\phi)}^{\frac{L_x}{l_B}-\frac{2\pi}{L_y}(mN_y+k_y+nN_\phi)} \mathrm{d}\xi \exp(-\xi^2) l_B = \int_{-\infty}^{\infty} \mathrm{d}\xi \exp(-\xi^2) l_B = \sqrt{\pi} l_B \tag{2.3.3}$$

We can restore the value of I in (2.3.2) to obtain the final result as expected:

$$\frac{1}{N_x} \sum_{m=-[\frac{N_x}{2}]}^{[\frac{N_x}{2}]} \exp\left(\frac{2\pi i m}{N_x}(k'_x - k_x)\right)\delta_{k'_y,k_y} = \delta_{k'_y,k_y}\delta_{k'_x,k_x} = \delta_{\mathbf{k},\mathbf{k}'} \tag{2.3.4}$$

As described earlier we will construct a Wannier function using the prescription:

$$\Phi_{\mathbf{m}}(\mathbf{r}) = \frac{1}{\sqrt{N_\phi}} \sum_k \Psi_{\mathbf{k}}(\mathbf{r}) \exp(-i\mathbf{k}\cdot\mathbf{m})$$

Where $\Psi_{\mathbf{k}}(\mathbf{r}) = \langle x, y|k\rangle$. With the normalization condition obtained for Bloch states (2.3.4) we ca easily check normalization of the newly constructed Wannier states:

$$\langle \Phi_{\mathbf{m}} | \Phi_{\mathbf{m}'} \rangle = \frac{1}{N_\phi} \sum_{k,k'} \underbrace{\int \mathrm{d}x\mathrm{d}y \Psi_{\mathbf{k}}(\mathbf{r})^\dagger \Psi_{\mathbf{k}'}(\mathbf{r})}_{\delta_{\mathbf{k},\mathbf{k}'}} \exp(-i(\mathbf{k}'\cdot\mathbf{m}' - \mathbf{k}\cdot\mathbf{m}))$$
$$= \frac{1}{N_\phi} \frac{(N_x N_y)}{(2\pi)^2} \int \mathrm{d}\mathbf{k} \exp(-i\mathbf{k}(\mathbf{m}' - \mathbf{m})) = \delta(\mathbf{m} - \mathbf{m}') \tag{2.3.5}$$

2.4 Examining the behavior of magnetic Wannier functions

Arbitrary magnetic Wannier has the form:

$$\Phi_{\mathbf{m}}(\mathbf{z}) = \frac{1}{\sqrt{N_\Phi}} \sum_{k_x,k_y} \frac{1}{\sqrt{N_x}} \sum_{m'=-[\frac{N_x}{2}]}^{[\frac{N_x}{2}]} \exp\left(i2\pi m' \frac{k_x}{N_x}\right) \left(\frac{1}{\sqrt{\pi} L_y l_B}\right)^{\frac{1}{2}} \times \tag{2.4.1}$$
$$\sum_n^{\mathbb{Z}} \exp\left(2\pi(m'N_y + k_y + nN_\Phi)\frac{x+iy}{L_y} - \frac{\pi L_x}{N_\Phi L_y}(m'N_y + k_y + nN_\Phi)^2\right)$$
$$\exp\left(-\frac{x^2}{2l_B^2}\right) \exp(-i\mathbf{k}\cdot\mathbf{m})$$

where $\mathbf{k} = (k_x, k_y)$. We can further simplify this expression by noticing that k_x only appears in the exponential function and the sum can be simply calculated.

$$\sum_{k_x=-[\frac{N_x}{2}]}^{[\frac{N_x}{2}]} \exp\left(i2\pi\frac{k_x}{N_x}(m'-m_x)\right) = N_x\delta_{m',m} \tag{2.4.2}$$

So the final expression is:

$$\Phi_{\mathbf{m}}(\mathbf{z}) = \frac{1}{N_y}\left(\frac{1}{\pi l_B^2\sqrt{2}}\right)^{\frac{1}{2}} \exp\left(-\frac{x^2}{2l_B^2}\right) \sum_{k_y=-[\frac{N_y}{2}]}^{[\frac{N_y}{2}]} \exp\left(-2\pi i\frac{k_y m_y}{N_y}\right) \tag{2.4.3}$$
$$\sum_{n}^{\mathbb{Z}} \exp\left(2\pi(m_xN_y + k_y + nN_\Phi)\frac{x+iy}{L_y} - \frac{\pi L_x}{N_\Phi L_y}(m_xN_y + k_y + nN_\Phi)^2\right)$$

Without loss of generality but for the sake of simplicity we will examine the function around the lattice point $\mathbf{m} = (0,0)$. We can redefine the $\mathbf{k}$ vector like $k_x \to 2\pi k_x/N_x$ and similarly with k_y so the whole expression simplifies. We can also set the cell unit length to $a = 1$ so that $l_B = \frac{1}{\sqrt{2\pi}}$, doing this every length will be expressed in units of unit-cell length:

$$\Phi_{0,0}(\mathbf{z}) = \frac{2^{\frac{1}{4}}\pi^{\frac{1}{4}}}{N_y}e^{-\pi x^2} \sum_{k_y=-\pi}^{\pi}\sum_{n}^{\mathbb{Z}} \exp\left(2\pi\left(\frac{k_y}{2\pi} + nN_x\right)z - \pi\left(\frac{k_y}{2\pi} + nN_x\right)^2\right) \tag{2.4.4}$$

The sum in the previous expression is hard to evaluate. Because we are interested in a long wavelength approximation we can turn to the thermodynamic limit (TD) and transform a sum into an integral over k_y:

$$\frac{N_y}{2\pi}\sum_{k_y=-\pi}^{\pi}\frac{2\pi}{N_y} \to \frac{N_y}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k_y \tag{2.4.5}$$

With TD approximation and using the fact that $L_i = l_B\sqrt{2\pi}N_i$ we arrive at the final expression for Wannier function around $\mathbf{m} = (0,0)$:

$$\Phi_{0,0}(\mathbf{z}) = \frac{1}{(2\pi)^{\frac{3}{4}}}e^{-\pi x^2}e^{\pi z^2}\sum_{n}^{\mathbb{Z}}\int_{-\pi}^{\pi}\mathrm{d}k_y \exp\left(-\left(\frac{k_y}{2\sqrt{\pi}} + \sqrt{\pi}nN_x - z\sqrt{\pi}\right)^2\right) \tag{2.4.6}$$

We can plot these functions in 2D contour plot and as line plot by slicing through x or y.

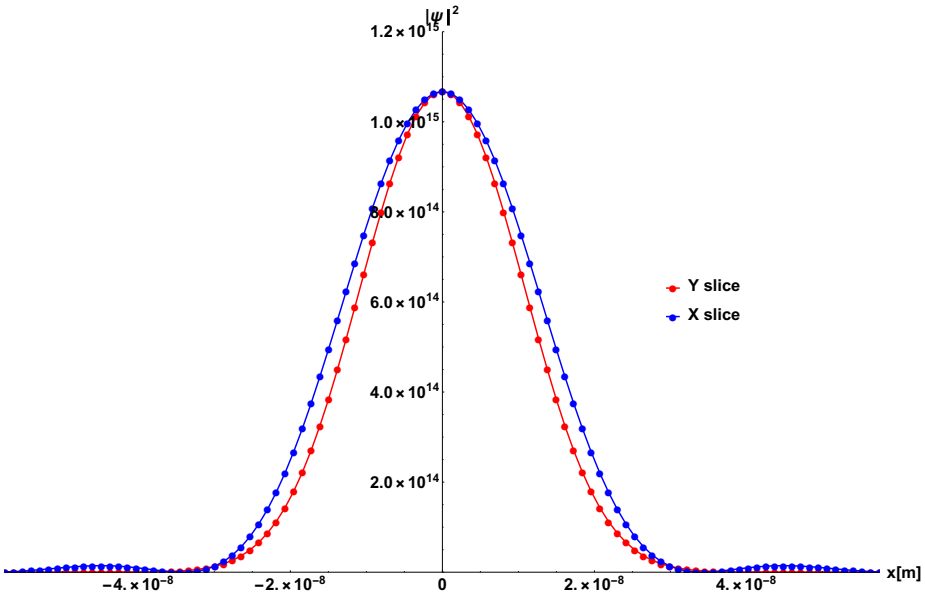


Figure 2.1: the exact function $x = 0$ (blue) and $y = 0$ (red) slice of the wave function

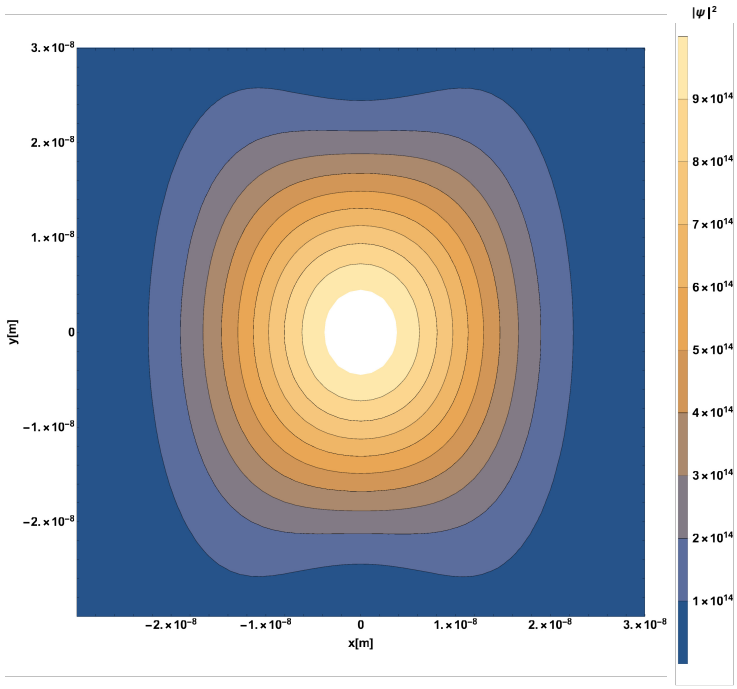


Figure 2.2: Contour plot of the exact wave function square moduo

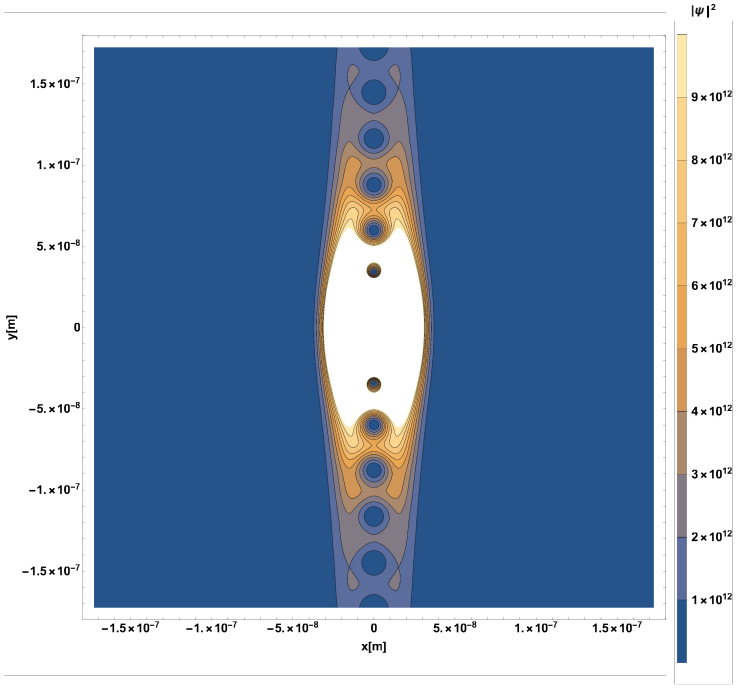


Figure 2.3: Contour plot of the exact wave function square moduo

We can see that the obtained wave function is indeed anisotropic as we expected. As can be seen from the figure 2.1, the function has a faster decay in x direction which can be expected from the Gaussian-like term in the original function. We are also expecting oscillations in y direction, a reminiscent of a plane-wave solution in Landau gauge. This behavior can be better seen in the graph below. In both images the used magnetic field was $B = 5\text{T}$, and unit cell size $a = 28.8\text{nm}$.

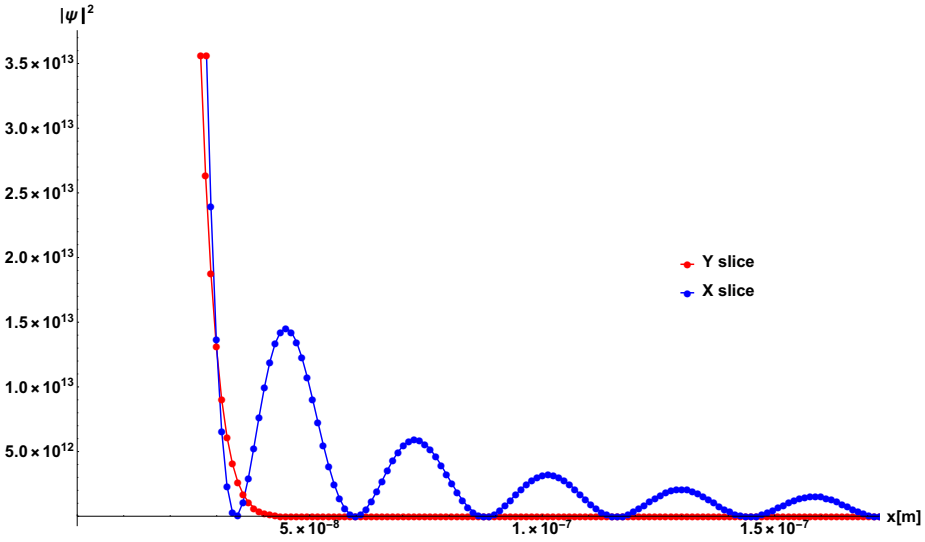


Figure 2.4: Detailed cross section in both direction, Y slice ($x = 0$) is in blue and X slice ($y = 0$) is in red

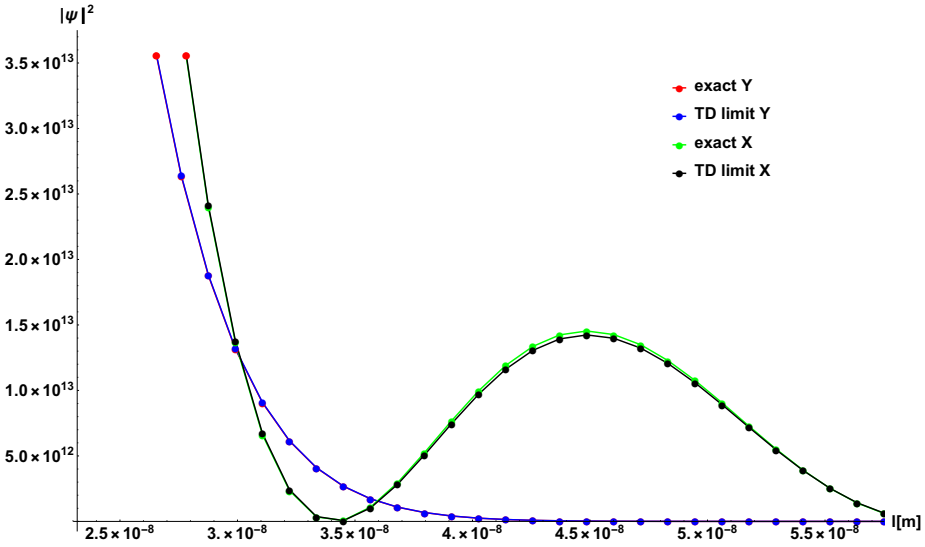


Figure 2.5: Comparison of TD limit and an exact function for X slice ($y = 0$) and Y slice ($x = 0$)

As we can see, the thermodynamic limit does not deviate much from the exact wave function.

Chapter 3

Interaction operator

In the previous chapter, we have constructed and examined the behavior of magnetic Wannier functions. Now we can use them to form the density operator and interaction operator afterwards.

3.1 Density operator

We will start by writing the density operator in Bloch basis and then transforming to magnetic Wannier:

$$\rho(\mathbf{q}) = \int \mathrm{d}\mathbf{r}\psi^\dagger(\mathbf{r})\psi(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}} \tag{3.1.1}$$

where $\psi(\mathbf{r}) = \frac{1}{\sqrt{N_\Phi}}\sum_{\mathbf{k}}\psi_{\mathbf{k}}b_{\mathbf{k}}$. On further expansion we have:

$$\rho(\mathbf{q}) = \frac{1}{N_\Phi}\sum_{\mathbf{k_1},\mathbf{k_2}} b^\dagger_{\mathbf{k_1}}b_{\mathbf{k_2}} \underbrace{\int \mathrm{d}\mathbf{r}\psi^\dagger_{k_1}(\mathbf{r})\psi_{k_2}(\mathbf{r})\exp\left(-i\frac{2\pi}{L_x}q_x x - i\frac{2\pi}{L_y}q_y y\right)}_{I} \tag{3.1.2}$$

The integral I can be separately evaluated:

$$I = \frac{1}{N_x}(\sqrt{\pi}L_y l_B)^{-1}\sum_{m,m'=-[\frac{N_x}{2}]}^{[\frac{N_x}{2}]}\exp\left(\frac{i2\pi}{N_x}(m'k_{2x} - mk_{1x})\right) \tag{3.1.3}$$

$$\sum_{n,n'}^{\mathbb{Z}}\int \mathrm{d}\mathbf{r}\exp\left(-\frac{x^2}{l_B^2} + \frac{2\pi x}{L_y}\left(mN_y + k_{1y} + nN_\Phi + m'N_y + k_{2y} + n'N_\Phi\right) - i\frac{2\pi}{L_x}q_x x\right)$$

$$\underbrace{\exp\left(\frac{2\pi i y}{L_y}\left(mN_y + k_{1y} + nN_\Phi - m'N_y - k_{2y} - n'N_\Phi - q_y\right)\right)}_{II}$$

$$\exp\left(-\frac{\pi L_x}{N_\Phi L_y}\left((mN_y + k_{1y} + nN_\Phi)^2 + (m'N_y + k_{2y} + n'N_\Phi)^2\right)\right)$$

The exponential function in II is a δ function, which we can evaluate:

$$II = \int_0^{L_y} \mathrm{d}y \exp\left(\frac{2\pi i y}{L_y}\left(mN_y + k_{1y} + nN_\Phi - m'N_y - k_{2y} - n'N_\Phi - q_y\right)\right) =$$
$$L_y \delta_{mN_y+k_{1y}+nN_\Phi, m'N_y+k_{2y}+n'N_\Phi} = L_y \delta_{m,m'}\delta_{n,n'}\delta_{k_{2y},k_{1y}+q_y} \tag{3.1.4}$$

Here the last term of delta function separation might at first look trivial, but we had to limit our self to the values of $q_y < N_y - k_{1y}$. This is the main assumption. Later on, it will be shown that integrals over q_y will have significant dumping so the integration can be done over the whole interval $[-\infty, \infty]$. The most relevant q_y are those for which $q_y \ll N_y$. This assumption will lead us toward effective one-dimensional reduction in the density operator for which any contribution $b^\dagger_{\mathbf{m}} b_{\mathbf{m}'}$ will have $m_x = m'_x$. Another point of view for this assumption will be given in the next section. We can use the obtained result to calculate the remaining integral I:

$$I = \frac{1}{N_x}(\sqrt{\pi} l_B)^{-1} \sum_{m=-[\frac{N_x}{2}]}^{[\frac{N_x}{2}]} \exp\left(\frac{2\pi i m}{N_x}(k_{2x} - k_{1x})\right) \tag{3.1.5}$$
$$\sum_n^{\mathbb{Z}} \exp\left(-\frac{\pi L_x}{N_\Phi L_y}\left((mN_y + k_{1y} + nN_\Phi)^2 + (mN_y + k_{1y} + q_y + nN_\Phi)^2\right)\right)$$
$$\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \mathrm{d}x \exp\left(-\frac{x^2}{l_B^2} + \frac{2\pi x}{L_y}\left(mN_y + k_{1y} + nN_\Phi + mN_y + k_{1y} + q_y + nN_\Phi\right) - i\frac{2\pi}{L_x} q_x x\right)$$

This integral can easily be solved by forming two symmetric Gaussian, and this can be achieved by the change of variables $k_{1y} \to k_{1y} - \frac{q_y}{2}$, now we have:

$$I = \frac{1}{N_x}(\sqrt{\pi} l_B)^{-1} \sum_{m=-[\frac{N_x}{2}]}^{[\frac{N_x}{2}]} \exp\left(\frac{2\pi i m}{N_x}(k_{2x} - k_{1x})\right) \tag{3.1.6}$$
$$\sum_n^{\mathbb{Z}} \exp\left(-\frac{\pi L_x}{N_\Phi L_y}\left((mN_y + k_{1y} - \frac{q_y}{2} + nN_\Phi)^2 + (mN_y + k_{1y} + \frac{q_y}{2} + nN_\Phi)^2\right)\right)$$
$$\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \mathrm{d}x \exp\left(-\frac{x^2}{l_B^2} + \frac{4\pi x}{L_y}\left(mN_y + k_{1y} + nN_\Phi\right) - i\frac{2\pi}{L_x} q_x x\right)$$

We can make the integrand full Gaussian and have for the final form:

$$I = \frac{1}{N_x} \sum_{m=-[\frac{N_x}{2}]}^{[\frac{N_x}{2}]} \sum_n^{\mathbb{Z}} \exp\left(\frac{2\pi i m}{N_x}(k_{2x} - k_{1x})\right) \tag{3.1.7}$$
$$\exp\left(-\frac{4l_B^2\pi^2 i}{L_x L_y}(mN_y + nN_\Phi + k_{1y})q_x - \frac{\pi^2 l_B^2}{L_x^2} q_x^2 - \frac{\pi^2 l_B^2}{L_y^2} q_y^2\right)$$
$$\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \mathrm{d}x \exp\left(-\left(\frac{x}{l_B} - \frac{l_B}{2}\left(\frac{4\pi}{L_y}(mN_y + nN_\Phi + k_{1y}) - \frac{2\pi i}{L_x} q_x\right)\right)^2\right)$$

The infinite sum over the $\mathbb{Z}$ makes the calculations a bit complicated, but we can incorporate it into the limits of the integral if we notice that:

$$e^{\frac{4l_B^2\pi^2 i}{L_xL_y}nN_\Phi q_x} = e^{-2\pi i n q_x} = 1 \tag{3.1.8}$$

Now that we only have n dependency inside the integral we can do change of variables $\xi = \frac{x}{l_B} - \frac{l_B}{2}\left(\frac{4\pi}{L_y}(mN_y + nN_\Phi + k_{1y}) - \frac{2\pi i}{L_x}q_x\right)$ and then the integral together with the sum become:

$$l_B\int_{-\infty}^{\infty} e^{-\xi^2}\mathrm{d}\xi = l_B\sqrt{\pi}$$

Final expression for the density of states in the Bloch basis is:

$$\begin{aligned}\rho(\mathbf{q}) =& \frac{1}{N_xN_\Phi}\sum_{\mathbf{k_1},\mathbf{k_2}} b^\dagger_{\mathbf{k_1}}b_{\mathbf{k_2}}\sum_{m=-[\frac{N_x}{2}]}^{[\frac{N_x}{2}]}\exp\left(\frac{2\pi i m}{N_x}(k_{2x}-k_{1x})\right) \\ &\exp\left(-\frac{4l_B^2\pi^2 i}{L_xL_y}(mN_y+k_{1y})q_x - \frac{\pi^2 l_B^2}{L_x^2}q_x^2 - \frac{\pi^2 l_B^2}{L_y^2}q_y^2\right)\delta_{k_{2y},k_{1y}+q_y}\end{aligned} \tag{3.1.9}$$

Now that we have a full density in the Bloch basis we can quickly go to the Wannier representation using:

$$b^\dagger_{\mathbf{k}} = \sum_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{k}}b^\dagger_{\mathbf{m}} \tag{3.1.10}$$

In this representation the density looks like:

$$\rho(\mathbf{q}) = \frac{1}{N_y}\sum_{k_{1y},m_{1y},m_{2y},m}\exp\left(\frac{2\pi i k_{1y}}{N_y}(m_{1y}-m_{2y})\right)\exp\left(-\frac{\pi i q_y}{N_y}(m_{1y}+m_{2y})\right) \tag{3.1.11}$$

$$\exp\left(-\frac{4l_B^2\pi^2 i}{L_xL_y}(mN_y+k_{1y})q_x - \frac{\pi^2 l_B^2}{L_x^2}q_x^2 - \frac{\pi^2 l_B^2}{L_y^2}q_y^2\right)b^\dagger_{\mathbf{m_1}}b_{\mathbf{m_2}}\delta_{m_{1x},m}\delta_{m_{1x},m_{2x}} \tag{3.1.12}$$

As we have seen before by going to the thermodynamic limit we are not deviating from the exact picture so that we will do the same here:

$$\begin{aligned}\rho(\mathbf{q}) = \frac{1}{2\pi}\sum_{k_{1y},m_{1y},m_{2y},m}&\exp\left(-\frac{iq_y}{2}(m_{1y}+m_{2y}) - imq_x - \frac{l_B^2}{4}(q_x^2+q_y^2) + \right. \\ &\left.\frac{iq_y}{2}(m_{1y}-m_{2y}) - \frac{iq_xq_y}{4\pi}\right)\underbrace{\int_{-\pi}^{\pi}\mathrm{d}k_{1y}\exp\left(ik_{1y}(m_{1y}-m_{2y}-\frac{q_x}{2\pi})\right)}_{III}\end{aligned} \tag{3.1.13}$$

Integral III can be evaluated separately to finally get the density operator (we have made a substitution $m \to m_x$). This operator is unfortunately not

Hermitian as a consequence of our approximation:

$$\rho(\mathbf{q}) = \frac{1}{2\pi}\sum_{\mathbf{m_1},\mathbf{m_2}} \exp\left(-iq_y m_{2y} - iq_x m_x - \frac{1}{8\pi}(q_x^2+q_y^2) - \frac{iq_xq_y}{4\pi}\right) \tag{3.1.14}$$
$$\frac{i}{\left(m_{1y}-m_{2y}-\frac{q_x}{2\pi}\right)} e^{i\pi(m_{1y}-m_{2y})}\left(\exp\left(i\frac{q_x}{2}\right)-\exp\left(-i\frac{q_x}{2}\right)\right)\delta_{m_{1x},m_{2x}}\delta_{m_x,m_{1x}} b^\dagger_{\mathbf{m_1}} b_{\mathbf{m_2}}$$

3.2 Density operator in Wu basis

In the work of Wu at al.[2] they have presented a new Bloch basis that we are also using. By deriving an expression for the density of states given in the paper, we can see what exactly the approximation in the previous section means physically. Here we work on the same torus $\mathbb{T} = [0, Lx] \times [0, Ly]$ pierced with N_Φ flux quantum. The LLL basis is spanned by:

$$|\mathbf{k}\rangle = \frac{1}{\sqrt{N_x}}\sum_{m=0}^{N_x-1} e^{i2\pi m k_x/N_x}\,|j = mN_y + k_y\rangle \tag{3.2.1}$$

where

$$\langle x,y|j\rangle = (\sqrt{\pi}L_y l_B)^{-1/2}\sum_{n}^{\mathbb{Z}} e^{2\pi(j+nN_\Phi)\frac{x+iy}{L_y} - \frac{\pi L_x}{N_\Phi L_y}(j+nN_\Phi)^2} e^{-\frac{x^2}{2l_B^2}} \tag{3.2.2}$$

It holds $|k_x+N_x, k_y+N_y\rangle = e^{-\frac{2\pi i k_x}{N_x}}|k_x,k_y\rangle$, we do not have a periodicity in y direction, every crossing of BZ in y direction will introduce additional phase factor. We want to calculate density operator:

$$\rho_{\mathbf{q}} = \int \mathrm{d}\mathbf{r}\,|\mathbf{r}\rangle\, e^{i\left(\frac{2\pi}{L_x}q_x x + \frac{2\pi}{L_y}q_y y\right)}\,\langle\mathbf{r}| \tag{3.2.3}$$

in the $\mathbf{k}$ basis so we have:

$$\langle\mathbf{k_1}|\rho_{\mathbf{q}}|\mathbf{k_2}\rangle = (\sqrt{\pi}L_y l_B N_x)^{-1}\sum_{n_1,n_2}^{\mathbb{Z}}\sum_{m_1,m_2=0}^{N_x-1}\int \mathrm{d}\mathbf{r}\, e^{-\frac{x^2}{l_B^2}} e^{i\frac{2\pi}{N_x}(m_2k_{2x}-m_1k_{1x})}$$
$$e^{x\left[\frac{2\pi}{L_y}(m_1N_y+k_{1y}+n_1N_\Phi)+\frac{2\pi i q_x}{L_x}+\frac{2\pi}{L_y}(m_2N_y+k_{2y}+n_2N_\Phi)\right]} \tag{3.2.4}$$
$$\underbrace{e^{\frac{2\pi i y}{L_y}(-m_1N_y-k_{1y}-n_1N_\Phi+q_y+m_2N_y+k_{2y}+n_2N_\Phi)}}_{I} e^{-\frac{\pi L_x}{N_\Phi L_y}\left[(m_2N_y+k_{2y}+m_2N_\Phi)^2+(m_1N_y+k_{1y}+m_1N_\Phi)^2\right]}$$

First, we will evaluate I as it is the only place y occurs:

$$\int_0^{L_y}\mathrm{d}y\, e^{\frac{2\pi i y}{L_y}(-m_1N_y-k_{1y}-n_1N_\Phi+q_y+m_2N_y+k_{2y}+n_2N_\Phi)} =$$
$$L_y\delta_{q_y+m_2N_y+k_{2y}+n_2N_\Phi,m_1N_y+k_{1y}+n_1N_\Phi} \tag{3.2.5}$$

We allow one of the $\mathbf{k}$'s to lie outside the BZ ($\mathbf{k_1}$ in our case) and then our delta function trivially decompose into:

$$\delta_{q_y+m_2N_y+k_{2y}+n_2N_\Phi,m_1N_y+k_{1y}+n_1N_\Phi} = \delta_{q_y+k_{2y},k_{1y}}\delta_{m_1,m_2}\delta_{n_1,n_2} \tag{3.2.6}$$

The fact that $\mathbf{k_1}$ can lie outside the BZ will introduce a phase shift whenever we cross the BZ so that the further care will be taken later. We can use this delta to write matrix element like:

$$\begin{aligned}\langle \mathbf{k_1}|\rho_{\mathbf{q}}|\mathbf{k_2}\rangle =&(\sqrt{\pi}l_B N_x)^{-1}\sum_{n_2}^{\mathbb{Z}}\sum_{m_2=0}^{N_x-1}\int \mathrm{d}x\, e^{-\frac{x^2}{l_B^2}} e^{i\frac{2\pi m_2}{N_x}(k_{2x}-k_{1x})} \\ & e^{x\left[\frac{2\pi}{L_y}(q_y+m_2N_y+k_{2y}+n_2N_\Phi)+\frac{2\pi i q_x}{L_x}+\frac{2\pi}{L_y}(m_2N_y+k_{2y}+n_2N_\Phi)\right]} \\ & \underbrace{e^{-\frac{\pi L_x}{N_\Phi L_y}\left[(m_2N_y+k_{2y}+m_2N_\Phi)^2+(q_y+m_2N_y+k_{2y}+n_2N_\Phi)^2\right]}}_{\dagger}\delta_{q_y+k_{2y},k_{1y}}\end{aligned} \tag{3.2.7}$$

Now we have integration over x that can be done by shifting a sum and simplifying the expression. First we shift $k_{2y} \to k_{2y} - \frac{q_y}{2}$ so that we have for the x integral:

$$\begin{aligned}&\int_0^{L_x}\mathrm{d}x\, e^{x\left[\frac{4\pi}{L_y}(m_2N_y+k_{2y}+n_2N_\Phi)+\frac{2\pi i}{L_x}q_x\right]}e^{-\frac{x^2}{l_B^2}} = \\ &\int_0^{L_x} e^{-\left(\frac{x}{l_B}-\frac{l_B}{2}\left(\frac{4\pi}{L_y}(m_2N_y+k_{2y}+n_2N_\Phi)+\frac{2\pi i}{L_x}q_x\right)\right)^2}\underbrace{e^{\frac{l_B^2}{4}\left(\frac{4\pi}{L_y}(m_2N_y+k_{2y}+n_2N_\Phi)+\frac{2\pi i}{L_x}q_x\right)^2}}_{\ddagger}\end{aligned} \tag{3.2.8}$$

Expanding † and combining with ‡ we have

$$e^{-\frac{\pi q_y^2}{2N_y^2}-\frac{\pi q_x^2}{2N_x^2}}e^{\frac{2\pi i q_x}{N_xN_y}(m_2N_y+k_{2y})} \tag{3.2.9}$$

in the expansion of ‡ we had a term with n_2 that was in a form $e^{2\pi i q_x n_2} = 1$. After these transformation we are left with an expression that only has n_2 dependence in the exp. The integral over x including infinite sum is:

$$\sum_{n_2}^{\mathbb{Z}}\int_0^{L_x} e^{-\left(\frac{x}{l_B}-\frac{l_B}{2}\left(\frac{4\pi}{L_y}(m_2N_y+k_{2y}+n_2N_\Phi)+\frac{2\pi i}{L_x}q_x\right)\right)^2} \tag{3.2.10}$$

we can do variable change $\xi = \frac{x}{l_B} - \frac{l_B}{2}\left(\frac{4\pi}{L_y}(m_2N_y + k_{2y} + n_2N_\Phi) + \frac{2\pi i}{L_x}q_x\right)$. Now n_2 only appears inside of integral limits and accounting for the infinite sum we get infinite limits of integration:

$$l_B\int_{-\infty}^{\infty}\mathrm{d}\xi\, e^{-\xi^2} = \sqrt{\pi}l_B \tag{3.2.11}$$

Collecting all the terms with m_2 and summing we have a delta function over the x component of momentum:

$$\frac{1}{N_x}\sum_{m_2=0}^{N_x-1} e^{m_2 i\frac{2\pi}{N_x}(k_{2x}-k_{1x}+q_x)} = \delta_{k_{1x},k_{2x}+q_x} \tag{3.2.12}$$

If we collect all the terms and re-shift momentum values back to the original state, finally we have for the matrix element:

$$\langle \mathbf{k_1}|\rho_{\mathbf{q}}|\mathbf{k_2}\rangle = e^{-\frac{\mathbf{q}^2}{8\pi}}e^{\frac{2\pi i q_x}{N_xN_y}(k_{2y}+\frac{q_y}{2})}\delta_{k_{1x},k_{2x}+q_x}\delta_{k_{1y},q_y+k_{2y}} \tag{3.2.13}$$

and density operator:

$$\rho_{\mathbf{q}} = e^{-\frac{\mathbf{q}^2}{8\pi}} e^{\frac{\pi i q_x q_y}{N_x N_y}} \sum_{\mathbf{k}}^{\mathrm{BZ}} e^{\frac{2\pi i q_x k_y}{N_x N_y}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}+\mathbf{q}} \tag{3.2.14}$$

We should bear in mind that if $\mathbf{k}+\mathbf{q}$ goes outside of BZ a phase factor appears

$$c_{\mathbf{k}+\mathbf{q}} = e^{\frac{2\pi i p_x n_y}{N_x}} c_{\mathbf{p}} \tag{3.2.15}$$

where we decomposed $\mathbf{k}+\mathbf{q} = \mathbf{p} + \mathbf{n}\cdot\mathbf{N}$. We can now see that assumption $q_y < N_y - k_{1y}$, that all momenta are from the IBZ, we have made in the previous section is a process of cutting the phase associated with going from one BZ to another. Also, by introducing this approximation, we are getting the previous expression for a density operator derived in (3.1.14).

3.3 Interaction operator

In the previous section, we have derived the density operator in the magnetic Wannier basis. Sadly because of the approximation made in order to separate a delta function, we have made a density non-hermitian. We would like to build an interaction operator that is Hermitian, so instead of a usual way we will build an interaction like $\sum_{\mathbf{q}} \rho^{\dagger}(\mathbf{q})\rho(\mathbf{q})$ so that it would be hermitian:

$$\int \mathrm{d}\mathbf{q}\rho^{\dagger}(\mathbf{q})\rho(\mathbf{q}) =$$
$$\frac{1}{4\pi^2}\int \mathrm{d}\mathbf{q} \sum_{\mathbf{m_1},\mathbf{m_2},\mathbf{m_1'},\mathbf{m_2'}} \exp\left(-iq_y(m_{2y}-m'_{2y}) - iq_x(m_x - m'_x) - \frac{1}{4\pi}(q_x^2+q_y^2)\right)$$
$$e^{i\pi(m_{1y}-m_{2y}-m'_{1y}+m'_{2y})}(-1)\left(e^{\frac{iq_x}{2}} - e^{-\frac{iq_x}{2}}\right)^2 \frac{1}{(m_{1y}-m_{2y}-\frac{q_x}{2\pi})(m'_{1y}-m'_{2y}-\frac{q_x}{2\pi})}$$
$$\delta_{m_{1x},m_{2x}}\delta_{m'_{1x},m'_{2x}}\delta_{m_x,m_{1x}}\delta_{m'_x,m'_{1x}} b^{\dagger}_{\mathbf{m_1}} b_{\mathbf{m_2}} b^{\dagger}_{\mathbf{m'_2}} b_{\mathbf{m'_1}} \tag{3.3.1}$$

We can represent this graphically by a lattice and site interactions:

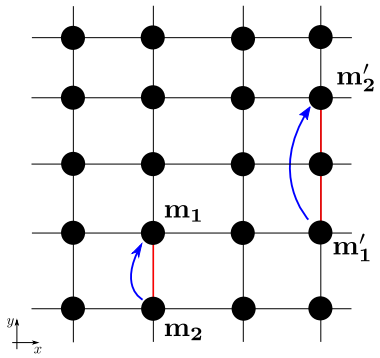


Figure 3.1: A schematical representation of general contribution to the interaction operator. Each density operator contribution $\sim b^\dagger_{m_1} b_{m_2}$ can be represented by a vertical line in the lattice structure (**red** line)

Detailed analysis of the matrix elements is given in Appendix A. All the relevant elements were analyzed and final model is proposed. Here we will just write the final expression for a Hamiltonian and discuss the results. The model Hamiltonian is given in this form:

$$\hat{H} = \sum_{\hat{n}} \Big[c^\dagger_{\hat{n}} c^\dagger_{\hat{n}} c_{\hat{n}} c_{\hat{n}} + \alpha c^\dagger_{\hat{n}} c^\dagger_{\hat{n}+\hat{x}} c_{\hat{n}} c_{\hat{n}+\hat{x}} + \beta c^\dagger_{\hat{n}} c^\dagger_{\hat{n}+\hat{y}} c_{\hat{n}} c_{\hat{n}+\hat{y}} + i\Delta (c^\dagger_{n+y} c_n - c^\dagger_n c_{n+y}) \\ (c^\dagger_{n-x} c_{n-x} + c^\dagger_{n+y-x} c_{n+y-x} - c^\dagger_{n+x} c_{n+x} - c^\dagger_{n+y+x} c_{n+y+x}) \Big] \tag{3.3.2}$$

where $\hat{n} = (n_x, n_y)$ is a vector labeling lattice and $\hat{x}$, $\hat{y}$ are unit vectors in x and y direction on a lattice.

Numerical studies were performed where $\alpha = \beta$ and chiral interaction was slowly introduced by a graduate increase of a δ parameter. Several system sizes were tested for a different number of particles so that filling of $\nu = \frac{1}{2}$ was satisfied. From the Laughlin state in a continuum, we were expecting a double degeneracy of the ground state. Numerical studies did not show the agreement for the number of degenerate states on a lattice. For example, for the system of size $N_x = 4$, $N_y = 4$ and $N = 8$ four-degeneracy was found. For the system of size $N_x = 3$, $N_y = 6$ and $N = 9$ the twelve-degeneracy was found. In these examples N_x and N_y are dimensions of the sample, the number of lattice points in x and y directions, and N is the number of bosons on a lattice.

Conclusion

In this thesis, we have tried to propose an effective lattice model Hamiltonian for bosons at $\nu = \frac{1}{2}$. We have worked in an anisotropic gauge[9,2] creating the magnetic Wannier function basis from which the density and interaction operator were constructed. In this process long wavelength approximation was introduced which enabled us to get the wire-like structures although this approximation is natural and the most simple one, the ensuing density was not Hermitian, and we had to impose further constraints on the interaction operator to make it Hermitian. Only the nearest neighbor elements were taken into consideration for the final model. Matrix elements that were included showed a similar behavior as in the previous literature.[10] The proposed model Hamiltonian was numerically diagonalized, but no evidence of Laughlin state at half filling was found. These findings bring us to the conclusion that a simple one-dimensional reduction scheme developed in this thesis that would produce Laughlin state on a lattice is not possible at the moment.

Appendix A

Analysis of a interaction operator

In the last section, we have derived the expression for an interaction operator in magnetic Wannier basis and gave a visual representation of a matrix element. Here we will give a detailed analysis of the most prominent elements and try to propose a model Hamiltonian.

A.1 On site interaction

The general element of this class is given on a picture below.

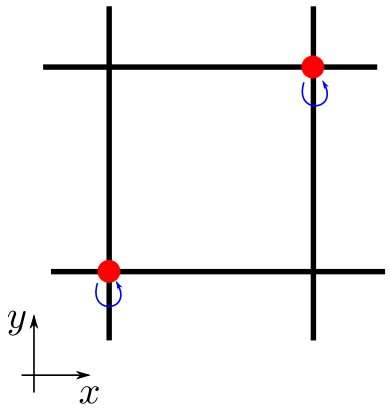


Figure A.1: onsite interaction

We will analyze the most general case of onsite interaction for which we

have $m_{1y} = m_{2y} \neq m'_{1y} = m'_{2y}$ and $m_x \neq m'_x$:

$$\frac{1}{4\pi^2}\int \mathrm{d}\mathbf{q}\exp\left(-iq_y(m_{2y}-m'_{2y}) - iq_x(m_x-m'_x) - \frac{1}{4\pi}(q_x^2+q_y^2)\right)$$
$$(-1)\left(e^{\frac{iq_x}{2}} - e^{-\frac{iq_x}{2}}\right)^2 \frac{1}{q_x^2} \tag{A.1.1}$$

we can write $\left(e^{\frac{iq_x}{2}} - e^{-\frac{iq_x}{2}}\right)^2 = -4\sin^2(\frac{q_x}{2})$ and use the approximation for small values of q_x, so $\sin(\alpha) \approx \alpha$. When we do the approximation and the integral over $\mathbf{q}$ we have:

$$e^{-\pi(m_{2y}-m'_{2y})^2} e^{-\pi(m_x-m'_x)^2} \tag{A.1.2}$$

From this result, we can see that repulsive interaction between sites falls off very quickly.

A.2 Onsite and hopping

This is one of the more interesting elements, and the general scheme is represented on a figure below:

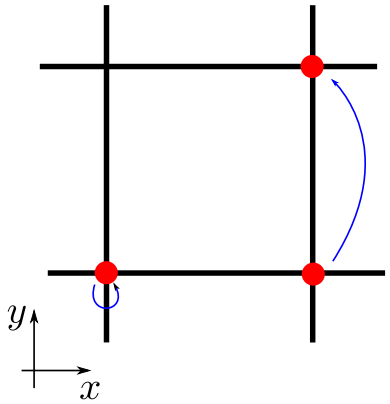


Figure A.2: Onsite and hop interaction element

In this case we have $m_{1y} = m_{2y} \neq m'_{1y} \neq m'_{2y}$ and $m_x \neq m'_x$. Here we made an approximation that $\frac{q_x}{2\pi} \ll 1$, by doing this we also keep the interaction operator Hermitian:

$$\frac{1}{4\pi^2}\int \mathrm{d}\mathbf{q}\exp\left(-iq_y(m_{2y}-m'_{2y}) - iq_x(m_x-m'_x) - \frac{1}{4\pi}(q_x^2+q_y^2)\right)e^{i\pi(-m'_{1y}+m'_{2y})} \tag{A.2.1}$$

$$\left(e^{\frac{iq_x}{2}} - e^{-\frac{iq_x}{2}}\right)^2 \frac{2\pi}{q_x(m'_{1y}-m'_{2y})}$$

We can do integration over y as before, but now we do not need to approximate for small q_x. Over q_x we have:

$$\int_{-\infty}^{\infty} \mathrm{d}q_x \exp\left(-\frac{q_x^2}{4\pi} - iq_x(m_x - m_x')\right) \sin^2\left(\frac{q_x}{2}\right) \frac{1}{q_x}$$

We can rewrite $\exp(iq_x(m_x - m_x'))$ over sin and cos, and see that real part with cos is an odd function on a symmetric interval so it will be zero and we are left with the purely imaginary solution. The integral we are trying to solve now is:

$$\int_{-\infty}^{\infty} \mathrm{d}q_x \exp\left(-\frac{q_x^2}{4\pi}\right) \sin(q_x(m_x - m_x')) \sin^2\left(\frac{q_x}{2}\right) \frac{1}{q_x} \tag{A.2.2}$$

Now we can transform this product of sin function, so we lose power

$$\begin{aligned}\sin(q_x(m_x - m_x')) \sin^2\left(\frac{q_x}{2}\right) = \frac{1}{4} \big[&\sin(q_x(1 - (m_x - m_x'))) + 2\sin(q_x(m_x - m_x')) \\ &- \sin(q_x(1 + m_x - m_x'))\big]\end{aligned}$$

After the transformation, we have integrals that can be solved. These integrals have form:[11]

$$\int_0^{\infty} x^{\mu-1} e^{-\beta x^2} \sin(\gamma x)\mathrm{d}x = \frac{\gamma e^{-\frac{\gamma^2}{4\beta}}}{2\beta^{\frac{\mu+1}{2}}} \Gamma\left(\frac{1+\mu}{2}\right) {}_1F_1\left(1 - \frac{\mu}{2}; \frac{3}{2}; \frac{\gamma^2}{4\beta}\right) \tag{A.2.3}$$

with conditions $\mathrm{Re}\{\beta\} > 0, \mathrm{Re}\{\mu\} > -1$. We can apply this to our integrals:

$$\begin{aligned}&-\frac{i\pi}{2}(1 - (m_x - m_x'))e^{-\pi(1-(m_x-m_x'))^2} {}_1F_1\left(1; \frac{3}{2}; \pi(1 - (m_x - m_x')^2\right) - \\ &\pi i(m_x - m_x')e^{-\pi(m_x-m_x')^2} {}_1F_1\left(1; \frac{3}{2}; \pi(m_x - m_x')^2\right) - \\ &-\frac{i\pi}{2}(1 - (m_x - m_x'))e^{-\pi(1+(m_x-m_x'))^2} {}_1F_1\left(1; \frac{3}{2}; \pi(1 + (m_x - m_x')^2\right)\end{aligned} \tag{A.2.4}$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind. We can verify that this is the solution to the integral by doing numerical integration and comparing with this solution.

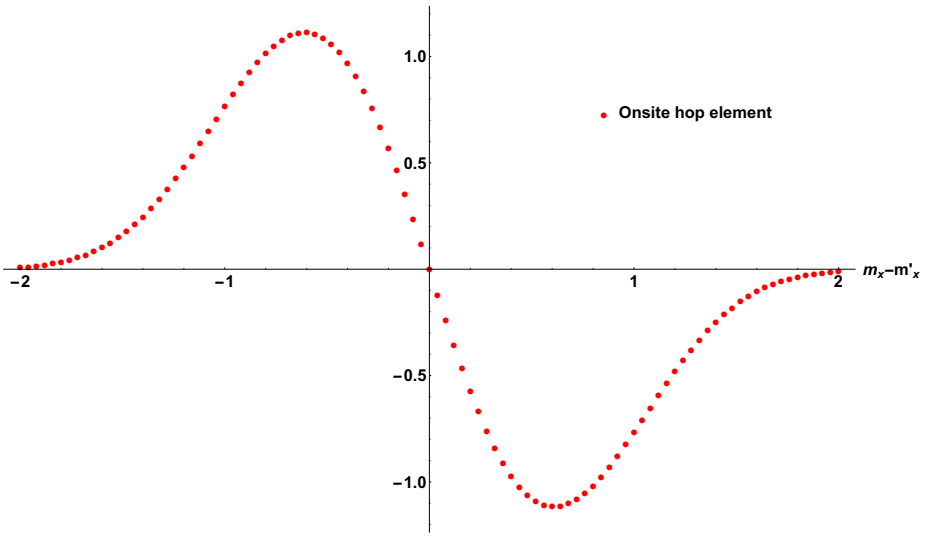


Figure A.3: magnitude of on-site hop element

We have a complex amplitude for interaction, and we can see that the sign changes depending on the direction of hopping which can indicate a possible chirality. Also, it can be seen from the graphics above that all the relevant contributions are from the nearest neighbor cells (x-axis is difference $m_x - m'_x$). We can see that we have the same sign for upward hopping on the right as a left hopping downward. This result has somewhat complicated form, if in the original integral we rewrite the part with $\sin^2$ like $\frac{\sin^2\left(\frac{q_x}{2}\right)}{q_x} \approx \frac{\sin\left(\frac{q_x}{2}\right)q_x}{2q_x} = \frac{\sin\left(\frac{q_x}{2}\right)}{2}$ and then do the integration we get

$$\frac{i\pi e^{-\frac{\pi}{4}}}{m'_{1y} - m'_{2y}} e^{-\pi(m_{2y}-m'_{2y})^2} e^{-\pi(m_x-m'_x)^2} \sinh(\pi(m_x - m'_x))$$

By doing this, we got a simpler form from which this can be directly seen. We also see that the imaginary phase that particle acquires while hops depend on the occupancy of neighboring cells, let us picture the general case:

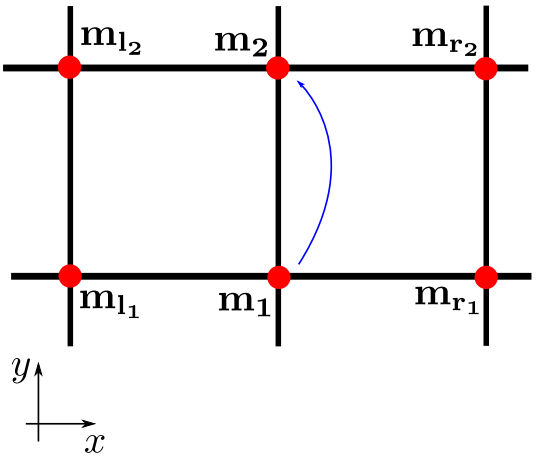


Figure A.4: Scheme for hopping element with the neighboring sites on which its amplitude depends

The dependency of hopping phase on occupancy is:

$$\sim i(n_{\mathbf{r_1}} + n_{\mathbf{r_2}} - n_{\mathbf{l_1}} - n_{\mathbf{l_2}}) \tag{A.2.5}$$

This is one of the main results in this section, and we will use it latter to construct the model Hamiltonian.

A.3 Double hop

Here we will examine the double hopping case represented on a scheme below:

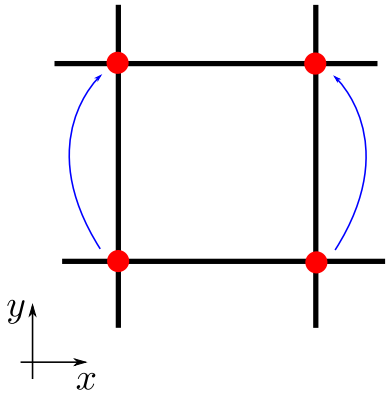


Figure A.5: Scheme of a double hop element

In this case we have $m_{1y} \neq m_{2y} \neq m'_{1y} \neq m'_{2y}$ and we will made an

approximation that $\frac{q_x}{2\pi} \ll 1$:

$$\frac{1}{4\pi^2}\int \mathrm{d}\mathbf{q}\exp\left(-iq_y(m_{2y}-m'_{2y})-iq_x(m_x-m'_x)-\frac{1}{4\pi}(q_x^2+q_y^2)\right) \tag{A.3.1}$$

$$e^{i\pi(m_{1y}-m_{2y}-m'_{1y}+m'_{2y})}\frac{q_x^2}{(m_{1y}-m_{2y})(m'_{1y}-m'_{2y})} =$$

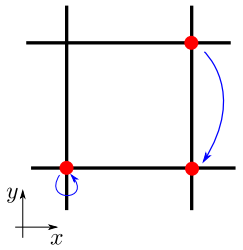
$$2\pi e^{-\pi(m_{2y}-m'_{2y})^2}e^{-\pi(m_x-m'_x)^2}e^{i\pi(m_{1y}-m_{2y}-m'_{1y}+m'_{2y})}\frac{(1-2\pi(m_x-m'_x)^2)}{(m_{1y}-m_{2y})(m'_{1y}-m'_{2y})}$$

Here we see that exponential dumping kills hops that end far apart. We can also see that if two hops happen over an odd difference of cells the interaction would be attractive otherwise is repulsive. However, these higher order hopping terms are dumped because of the Gaussian term.

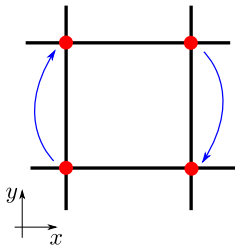
A.4 Magnitude comparison

We have examined the three types of elements in the interaction operator. Here we will compare their magnitude and try to select the ones that will be in a final model.

We did a numerical calculation for each element without any approximation in their full form for the comparison to be complete. For onsite and double hop real part is plotted, while for the onsite and hop imaginary part is plotted. We are considering the following situation best depicted in the figures below:



(a) Onsite hop configuration



(b) Double hop configuration

Figure A.6

These images represent the following conditions $m_{2y} - m'_{2y} = 0$, $m_{1y} - m_{2y} = 1$ and $m'_{1y} - m'_{2y} = 1$.

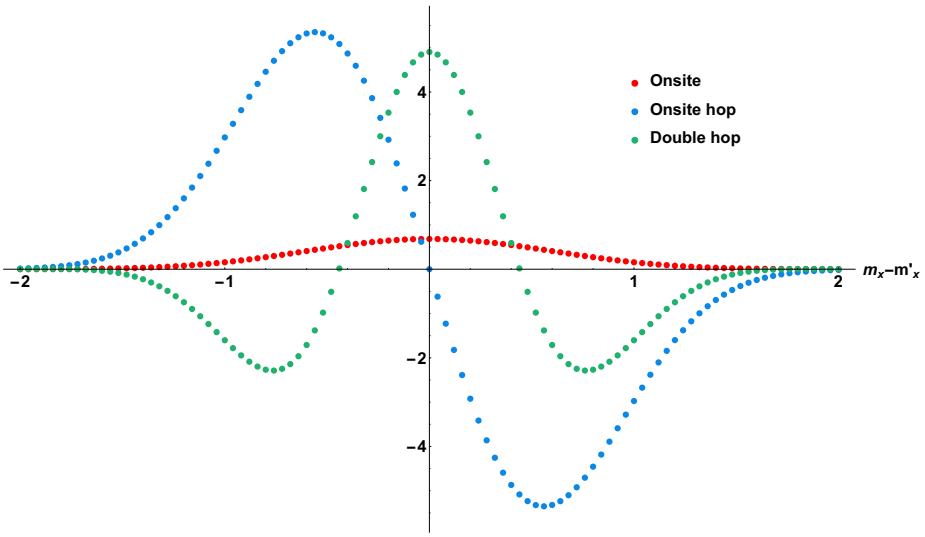
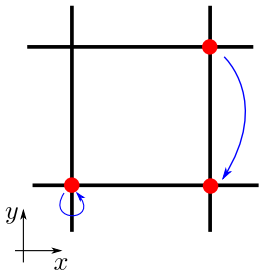
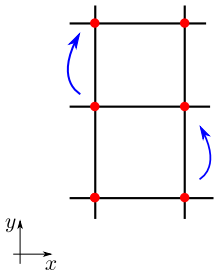


Figure A.7: Magnitude comparison for real part of onsite and double hop, imaginary part for onsite and hop element

Another interesting case is represented on a pictures below:



(a) Onsite hop element configuration



(b) Double hop element configuration

Figure A.8

In this configuration the setup is following $m_{2y} - m'_{2y} = 0$, $m_{1y} - m_{2y} = 1$ and $m'_{1y} - m'_{2y} = -1$.

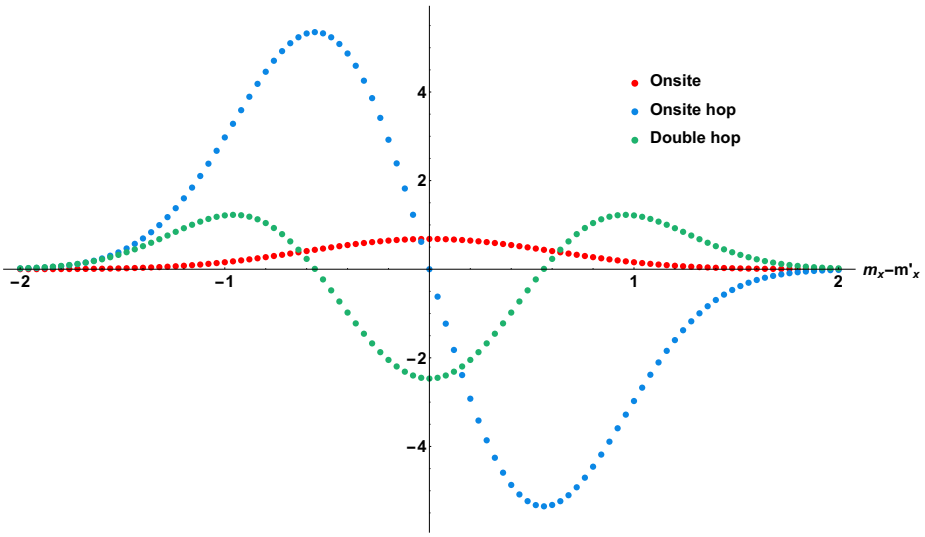


Figure A.9: Magnitude comparison for real part of onsite and double hop, imaginary part for onsite and hop element

As it can be seen from these results, our model will consist of a lattice with bosons distributed in such a way that each site will have a repulsive interaction between each other similar as electrons trapped in a magnetic field and the hopping term would introduce the wanted chirality as in a magnetic field. The double hopping terms could be incorporated as a way to disrupt the perfect lattice. It should be noted that higher order double hopping terms were analyzed and no significant terms were found. In the process of approximating the Hamiltonian, we have indeed broken some symmetries of the original problem. This can lead to erratic behavior of nonlocal matrix elements. Because of this, we are restricting our self to the nearest neighbor elements. Proposed model could be given in a form with parameters that could tune the interactions and energies:

$$\hat{H} = \sum_{\hat{n}} \Big[c^\dagger_{\hat{n}} c^\dagger_{\hat{n}} c_{\hat{n}} c_{\hat{n}} + \alpha c^\dagger_{\hat{n}} c^\dagger_{\hat{n}+\hat{x}} c_{\hat{n}} c_{\hat{n}+\hat{x}} + \beta c^\dagger_{\hat{n}} c^\dagger_{\hat{n}+\hat{y}} c_{\hat{n}} c_{\hat{n}+\hat{y}} + i\Delta(c^\dagger_{n+y} c_n - c^\dagger_n c_{n+y}) \\ (c^\dagger_{n-x} c_{n-x} + c^\dagger_{n+y-x} c_{n+y-x} - c^\dagger_{n+x} c_{n+x} - c^\dagger_{n+y+x} c_{n+y+x}) \Big] \tag{A.4.1}$$

where $\hat{n} = (n_x, n_y)$ is a vector labeling lattice and $\hat{x}$, $\hat{y}$ are unit vectors in x and y direction on a lattice.

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